NEHRU COLLEGE OF ENGINEERING AND RESEARCH CENTRE
(NAAC Accredited)
(Approved by AICTE, Affiliated to APJ Abdul Kalam Technological University, Kerala)

DEPARTMENT OF ELECTRONICS & COMMUNICATION ENGINEERING

COURSE MATERIALS

MA 102 DIFFERENTIAL EQUATIONS

VISION OF THE INSTITUTION

To mould true citizens who are millennium leaders and catalysts of change through excellence in education.

MISSION OF THE INSTITUTION

NCERC is committed to transform itself into a center of excellence in Learning and Research in Engineering and Frontier Technology and to impart quality education to mould technically competent citizens with moral integrity, social commitment and ethical values.

We intend to facilitate our students to assimilate the latest technological know-how and to imbibe discipline, culture and spiritually, and to mould them in to technological giants, dedicated research scientists and intellectual leaders of the country who can spread the beams of light and happiness among the poor and the underprivileged.

ABOUT DEPARTMENT

- Established in: 2002
- Course offered : B.Tech in Electronics and Communication Engineering

  M.Tech in VLSI
- Approved by AICTE New Delhi and Accredited by NAACAffiliated to the University of Dr. A P J Abdul Kalam Technological University.

DEPARTMENT VISION

Providing Universal Communicative Electronics Engineers with corporate and social relevance towards sustainable developments through quality education.

DEPARTMENT MISSION

1) Imparting Quality education by providing excellent teaching, learning environment.

2) Transforming and adopting students in this knowledgeable era, where the electronic gadgets (things) are getting obsolete in short span.

3) To initiate multi-disciplinary activities to students at earliest and apply in their respective fields of interest later.

4) Promoting leading edge Research & Development through collaboration with academia & industry.

PROGRAMME EDUCATIONAL OBJECTIVES

Graduates of Mechatronics Engineering will:

PEO1: Graduates shall have the ability to work in multidisciplinary environment with good professional and commitment.

PEO2: Graduates shall have the ability to solve the complex engineering problems by applying electrical, mechanical, electronics and computer knowledge and engage in lifelong learning in their profession.

PEO3: Graduates shall have the ability to lead and contribute in a team entrusted with professional social and ethical responsibilities.

PEO4: Graduates shall have ability to acquire scientific and engineering fundamentals necessary for higher studies and research.

PROGRAM OUTCOMES (POS)

Engineering Graduates will be able to:

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

PROGRAM SPECIFIC OUTCOMES (PSO)

PSO 1: Design and develop Mechatronics systems to solve the complex engineering problem by integrating electronics, mechanical and control systems.

PSO 2: Apply the engineering knowledge to conduct investigations of complex engineering problem related to instrumentation, control, automation, robotics and provide solutions.

COURSE OUTCOMES

CO1	Solve the convergent test in mathematical series
CO2	Acquire the basic knowledge about three dimensional spaces and integral calculus of functions of more than one variables
CO3	Understand about partial derivatives and its applications
CO4	Solve problems in calculus of vector valued functions
CO5	. Apply multiple integrals to find area and volume
CO6	Evaluate surface and volume integrals

MAPPING OF COURSE OUTCOMES WITH PROGRAM OUTCOMES

	PO1	**PO2**	**PO3**	**PO4**	**PO 5**	**PO6**	**PO7**	**PO8**	**PO9**	**PO10**	**PO11**	**PO12**
CO1	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO2	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO3	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO4	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO5	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**
CO6	**3**	**3**	**3**	**3**	**2**	**1**			**1**	**2**		**2**

Note: H-Highly correlated=3, M-Medium correlated=2, L-Less correlated=1

	PSO1	**PSO2**	**PSO3**
CO1	2	1	
CO2	2	1	
CO3	1	1	
CO4	1	1	
CO5	1	1	

CO6	1	1	

SYLLABUS

COURSE NO.	COURSE NAME	L-T-P-CREDITS	YEAR OF INTRODUCTION
MA102	DIFFERENTIAL EQUATIONS	3-1-0-4	2015

COURSE OBJECTIVES

This course introduces basic ideas of differential equations, both ordinary and partial, which are widely used in the modelling and analysis of a wide range of physical phenomena and has got applications across all branches of engineering. The course also introduces Fourier series which is used by engineers to represent and analyse periodic functions in terms of their frequency components.

Syllabus

Homogeneous linear ordinary differential equation, non-homogeneous linear ordinary differential equations, Fourier series, partial differential equation, one dimensional wave equation, one dimensional heat equation.

EXPECTED OUTCOME

At the end of the course students will have acquired basic knowledge of differential equations and methods of solving them and their use in analysing typical mechanical or electrical systems. The included set of assignments will familiarise the students with the use of software packages for analysing systems modelled by differential equations.

TEXT BOOKS

1. Erwin Kreyszig: Advanced Engineering Mathematics, 10th ed. Wiley
2. A C Srivastava, P K Srivasthava, Engineering Mathematics Vol 2. PHI Learning Private Limited, New Delhi.

REFERENCES:

1. Simmons: Differential Equation with Applications and its historical Notes,2e McGrawHill Education India 2002
2. Datta, Mathematical Methods for Science and Engineering. CengageLearing,1st. ed
3. B. S. Grewal. Higher Engineering Mathematics, Khanna Publishers, New Delhi.
4. N. P. Bali, Manish Goyal. Engineering Mathematics, Lakshmy Publications
5. D. W. Jordan, P Smith. Mathematical Techniques, Oxford University Press, 4th Edition.
6. C. Henry Edwards, David. E. Penney, Differential Equations and Boundary Value

QUESTION BANK

MODULE –I

1. Solve the ODE $y''' + y = 0$
2. Find the characteristic roots of the ODE $y'' + 2y' + 5y = 0$
3. State existence and uniqueness theorem of initial value problem(Picard's Theorem)
4. State existence and uniqueness of solutions
5. State Super position principle
6. Explain linear independence and dependence of solutions
7. Solve the ODE$y^{v} - 3y^{iv} + 3y^{iii} - y^{ii} = 0$
8. Find an ODE for which the functions$e^{x}, e^{-x}, cosx, sinx$ form a basis of solutions
9. Check whether the functions $(x+1), (x+2), x$ are linearly independent or not
10. If $e^{-2x}, xe^{-2x}, x^2e^{-2x}$forms a basis ,then find the corresponding ODE
11. Solve $y'' + 9\pi^2 y = 0$
12. Solve $4y'' - 20y' + 25y = 0$
13. Find an ODE for the given basis $e^{\sqrt{3}x}$, $xe^{\sqrt{3}x}$.
14. Verify by substitution that the given functions form a basis. Solve the given problem.
    $y'' + 2y' + 2y = 0$, $e^{-x}\cos x$, $e^{-x}\sin x$ y(0) = 1 , , y' (0) = -1 .
15. Solve $xy'' + 2y' + xy = 0$ if $y_1 = \frac{cosx}{x}$is a solution by reducing the differential equation into lower order?
16. Find a basis of solution of the ODE $(x^2 - x)y'' - xy' + y = 0,$ given$y_1 = x$ is a solution?
17. Verify that y=x^3 is a solution of the ODE $x^2y'' - 5xy' + 9y = 0$ and find the second solution by reducing it to lower order
18. Verify that y=x^{-1} is a solution of the ODE $2x^2y'' + 3xy' - y = 0$ and find the second solution by reducing it to lower order
19. Solve $(D^2 + 1)y = 0$ subject to y (0) =3,$y'(0) = \frac{-1}{2}$.
20. Solve $y'' + 4y' + 5y = 0$with initial conditionsy (0)=2 ,$y'(0) = -5$.

MODULE II

1. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{2x}$.
2. Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = x^2$.
3. Solve by method of variation of parameters $\frac{d^2y}{dx^2} + y = xsinx$
4. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x sinx$.
5. Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 + 2logx$.
6. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = sin3xsin2x$.
7. Find the particular integral of $(D^2 + 4D + 1)y = e^x sin3x$
8. Solve $(D^3 + 8)y = sinxcosx + e^{-2x}$.
9. Solve $y'' + y = tanx$ by the method of variation of parameters.
10. Solve $x^3\frac{d^3y}{dx^3} - 2x^2\frac{d^2y}{dx^2} + 2y = \frac{1}{x}$.
11. Solve $(D^2 + 2D - 3)y = e^x cosx$
12. Find the particular integral of $(D^2 - 2D + 1)y = xe^x$
13. Solve by the method of variation of parameters $(D^2 + 4)y = tan2x$
14. Solve $(D + 1)^2 = x^2e^x$
15. Solve the differential equation $(x^3D^3 + 3x^2D^2 + xD + 1)y = x + logx$
16. Solve the differential equation $(D^2 + 1)y = x^2e^x + sinx$
17. Solve $(x + 1)^2y'' + (x + 1)y' - y = 2\sin\log(x + 1)$
18. Find the particular integral of $(D^4 - m^4)y = sinmx$
19. Use variation of parameters to solve $y'' + y = secx$
20. Solve $(D^2 - 2D + 5)y = e^{2x}sinx$
21. Solve $(x + 1)^2\frac{d^2y}{dx^2} + (x + 1)\frac{dy}{dx} = (2x + 3)(2x + 4)$
22. Solve $x^3\frac{d^3y}{dx^3} - 2x^2\frac{d^2y}{dx^2} + 2y = 10(x + \frac{1}{x})$.
23. Solve $y'' - 4y' + 5y = \frac{e^{2x}}{sinx}$ by the method of variation of parameters.

24. Find the particular integral of $y''+y=\sin x$.

MODULE –lll

Find the Fourier coefficient a$_n$ for the function $f(x) = 1 + |x|\ defined\ in - 3 < x < 3$

Find the Fourier series representation of $f(x) = xsinx$ periodic with period $2\pi\ defined - \pi < x < \pi$

Find the Fourier series expansion of $f(x) = e^{-x}\ in - c < x < c$

Express $f(x) = x$ as a Fourier series in the interval $-\pi < x < \pi$

Obtain the Fourier series for the function $f(x) = \begin{cases} 1 + \frac{2x}{\pi} & -\pi \leq x \leq 0 \\ 1 - \frac{2x}{\pi} & 0 \leq x \leq \pi \end{cases}$

Obtain the Fourier series to represent the function $f(x) = |\sin x| - \pi < x < \pi$

Develop the Fourier series of $f(x) = x^2 \qquad - 2 < x < 2$

Obtain the Fourier series for the function $f(x) = \begin{cases} x & 0 < x < 1 \\ 1 - x & 1 < x < 2 \end{cases}$

Develop the Fourier series of $f(x) = e^{-x} \qquad - l < x < l$

Obtain the Fourier series for the function $f(x) = \begin{cases} -\frac{\pi}{4} & -\pi \leq x \leq 0 \\ \frac{\pi}{4} & 0 \leq x \leq \pi \end{cases}$

Obtain the Fourier series for the function $f(x) = \begin{cases} -1 + x & -\pi \leq x \leq 0 \\ 1 + x & 0 \leq x \leq \pi \end{cases}$

Develop the Fourier sine series of $f(x) = x\ in\ 0 < x < \pi$

Develop the Fourier cosine series of $f(x) = cosx\ in\ 0 < x < \frac{\pi}{2}$

Develop the Fourier sine series of $f(x) = \begin{cases} x & 0 < x < 2 \\ 4 - x & 2 < x < 4 \end{cases}$

Obtain the half range Fourier sine series for the function $f(x) = e^x \quad 0 < x < 2$

Find the half range cosine series for the function $f(x) = x^2 \quad 0 \leq x \leq \pi$

Find the half range cosine series for the function $f(x) = x(\pi - x)0 < x < \pi$

Find the half range cosine series for the function $f(x) = xsin\ x \quad 0 < x < \pi$

Find the half range cosine series for the function $f(x) = x^2 \quad 0 \leq x \leq c$

Find the half range cosine series for the function $f(x) = x \quad 0 \leq x \leq l$

Find the half range sine series for the function $f(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 < x < 2 \end{cases}$

Find the Fourier series of the periodic function f(x) of period 4, where f(x)=$\begin{matrix} 2 & -2 \leq x \leq 0 \\ x & 0 \leq x \leq 2 \end{matrix}$

duce that (i) 1+$\frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \ldots = \frac{\pi^2}{8}$ (ii) 1-$\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \ldots = \frac{\pi}{4}$

MODULE –IV

Form the partial differential equation from $z=xg(y)+yf(x)$

Solve $(y-z)p+(x-y)q=(z-x)$

Solve the partial differential equation $(y^2+z^2)p-xyq+xz=0$

Form the partial differential equation by eliminating arbitrary function from $z=y^2+2f(\frac{1}{x}+\log y)$

Solve$p\sqrt{x}+q\sqrt{y}=\sqrt{z}$

Form the partial differential equation by eliminating arbitrary function from $x^2+y^2+z^2=f(xy)$

Solve$y^2zp+x^2zq=xy^2$

a) Solve $(y+zx)p-(x+yz)q=x^2-y^2$

) Find the differential equation of all spheres of fixed radius having their centers in XY-plane

a) Solve $(D^2-2DD'-15D'^2)z=12xy$

b)Find the particular integral of $(D^3-7DD'^2-6D'^3)Z=sin\,(x+2y)+e^{3x+y}$

a) Solve $\frac{\partial^3 Z}{\partial x^3}-2\frac{\partial^3 z}{\partial x^2\partial y}=5e^{3x}$-$7x^2y$

b) $Solve\ (x+y)zp+(x-y)zq=x^2+y^2$

a) Solve $\frac{\partial^3 Z}{\partial x^3}-4\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial x\partial y^2}=2\sin(3x+2y)$

b) Solve $\frac{\partial^2 Z}{\partial x^2}-\frac{\partial^2 Z}{\partial y^2}=\cos 2x\,cos\,3y$

a) Form the partial differential equation by eliminating a,b,c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

b) Solve p d e Solve $\frac{\partial^2 Z}{\partial x^2}+\frac{\partial^2 Z}{\partial x\partial y}-\frac{\partial^2 Z}{\partial y^2}=e^{2x+y}$

a) Solve $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

b) Solve $\frac{\partial^3 Z}{\partial x^3}-4\frac{\partial^3 z}{\partial x^2\partial y}+4\frac{\partial^3 z}{\partial x\partial y^2}=\cos(2x+y)$

a) Solve $(D^2-2DD'+D'^2)z=e^{x+2y}+x^3$

b)) Find the particular integral of Solve $\frac{\partial^3 Z}{\partial x^3}-7\frac{\partial^3 z}{\partial x^2\partial y}-6\frac{\partial^3 Z}{\partial y^3}=\sin(x+2y)$

a)Solve $(D^2+DD'-6D'^2)z=ysinx$

b) Solve $(mz-ny)p+(nx-lz)q=(ly-mx)$

MODULE –V

1.Write down the important assumptions when derive one dimensional wave equation

2.Solve$3u_x + 2u_y = 0\ with\ u(x,0) = 4e^{-x}\ \ by\ the\ method\ of\ seperation\ of\ variables.$

3. Solve one dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$with boundary conditions $u(0,t) = 0\ \ ,u(l.t) = 0$ for all t and initial conditions $u(x,) = f(x),\quad \left.\frac{\partial u}{\partial t}\right\}_{t=0} = g(x)$

4.A string of length 20 cm fixed at both ends is displaced from its position of equilibrium, by each of its points an initial velocity given by $\begin{cases} x & 0 < x \leq 10 \\ 20 - x & 10 \leq x \leq 20 \end{cases}$ $,x$ being the distance from one end. Determine the displacement at any subsequent time.

5. Solve one dimensional wave equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$using method of separation of variables when the separation constant $k < 0$

6. A tightly stretched string of length ‘a’ with fixed ends is initially in equilibrium position. Find the displacement $u(x,t)$ of the string if it is vibrating by giving each of its points a velocity $v_0 \sin(\frac{\pi x}{a})$

7. A transversely vibrating string of length ‘a’ is stretched between two points A and B. The initial displacement of each point of the string is zero and the initial velocity at a distance x from A is kx(a-x). Find the form of string at any subsequent time.

8. State one dimensional wave equation with boundary conditions and initial conditions for solving it.

9. Using the method of separation of variables solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u\ \ where\ u(x,0) = 3e^{-5x}$

10. A tightly stretched string of with fixed end points x=0 and x=l is initially in a position given by $y = y_0 sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest find the displacement y(x,t).

11. A tightly stretched string of with fixed end points x=0 and x=l is initially at rest in equilibrium position. If it is vibrating by giving each of its points a velocity is λx(l-x), find the displacement of the string at any distance x from one end at any time t.

12.Solve$u_x + u_y = 0\ using\ method\ of\ seperation\ of\ variables.$

13. A finite string of length L is fixed at both ends and is released from rest with a displacement f(x). What are the initial and boundary conditions involved in this problem.

14. A tightly stretched homogenous string of length 20cm with its fixed ends executes transverse vibrations. Motion starts with zero initial velocity by displacing the string into the form $f(x) = K(x^2 - x^3).\ Find\ the\ deflection\ u(x,t)\ at\ any\ time\ t$

15. Derive one dimensional wave equation.

16. Solve one dimensional wave equation by the method of separation of variables.

MODULE –VI

1.In heat equation $\frac{\partial u}{\partial t}$ =$\propto^2$ $\frac{\partial^2 u}{\partial x^2}$ what does $\propto^2$ indicate. State the boundary conditions for solving it.

2.Find the steady state temperature distribution in a rod of length 25cm, if the ends of the rod are kept at 20°c and 70°c.

3. A bar 10cm long with insulated sides has its ends A and B maintained at 30°c and 100°c respectively until steady state conditions prevail. The temperature at A is suddenly raised to 20°c and at the same time that of B is lowered to 40°c.Find the temperature distribution in the bar at time t.

4. A rod of 30cm long with insulated sides has its ends A and B maintained at 30°c and 90°c respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to zero temperature and kept so. Find the temperature distribution u(x,y) taking x=0 at A.

5. Solve $\frac{\partial u}{\partial t} = h\frac{\partial^2 u}{\partial x^2}$ subject to the conditions u(0,t)=u(1,t)=0 for t>0 and $u(x,0) = 3sin\, n\pi x \quad 0 < x < 1$

6. Find the steady state temperature distribution in a rod of length 20cm, if the ends of the rod are kept at 10°c and 70°c.

7. Find the temperature in a laterally insulated bar of length L whose ends are kept at temperature zero if the initial temperature is $f(x) = \begin{cases} x & 0 < x < L/2 \\ L - x & \frac{L}{2} < x < L \end{cases}$

8.A rod of length L has its ends A and B maintained at 0°c and 100°c respectively until steady state conditions prevail. If B is suddenly reduced to 0°c and maintained at 0°c, then find the temperature in the rod at a distance x from A at time t.

9. Solve one dimensional heat equation when k>0

10. Write down all possible solutions of one dimensional heat equation.

11. Derive one dimensional heat equation

12. Find the steady state temperature distribution in a rod of length 30cm, if the ends of the rod are kept at 20°c and 80°c.

13. Find the temperature in a laterally insulated bar of length 2cm whose ends are kept at temperature zero if the initial temperature is $f(x) = 100(2x - x^2)$

14. A rod of length l with insulated sides has its ends A and B maintained at 30°c and 90°c respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to zero temperature and kept so find the temperature distribution at a distance x at time t.

20/1/2016
Wednesday

Module 1.

Homogereous Equations. (Homogeneous Differential Equations.)

- Existance and Uniqueness of solutions for initial value problems.
- Homogeneous linear oridinary differential equations of second order.
- Homogeneous linear ordinary differential equations with constant coefficients, existance and uniqueness of solutions.
- Wronskain, homogeneous linear ordinary differential equations with constant coefficient of higher order.

Differential Equations.

An equation in which the differential coefficients occurs is called a differential equation.

eg:

① $\frac{d^2y}{dx^2} + 9y = 0$

② $\frac{dy}{dx} + yx^2 = x^2$

③ $\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} = 0$

④ $\frac{\partial^2 u}{\partial x^2 \partial y} + x\frac{\partial^2 u}{\partial y^2} = x^2$

Oridinary Differential Equations. (ODE)

A differential equation involving a single independent variable and hence only ordinary derivatives is called an oridnary differential equation. $\boxed{\text{or}}$ An ordinary differential equation (ODE) is an equation that contains one or several derivatives

of an unknown function $y(x)$.
In the above set of four differential equations the equations ① and ② are ODE's.

<u>Note:</u>
The ③ and ④ equations are partial differential equations ~~or~~ PDE's

<u>Order</u> of a ordinary linear differential equation is the order of the highest derivative present in that equation.

The <u>Degree</u> of a ordinary linear differential equation is the degree of the highest derivative occuring in it, after the equation has been expressed in a form free from radicals and fraction

eg: ① $y' = \cos x$
Order $= 1$ (y') , Degree $= 1$

② $y'' + 9y = e^{-2x}$
Order $= 2$, Degree $= 1$

③ $y' \cdot y''' - \frac{3}{2}(y'')^2$
Order $= 3$, Degree $= 1$ / degree of the highest derivati

- Non-homogeneous
$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots = g(x)$
C.S = C.F + P.I $(a_0 \neq 0)$
- If Homogeneous $(a_0 = 0)$ g(x) =
C.S = C.F + (complementary function, [illegible]

* <u>Linear Differential Equations</u> (LDE)

A LDE is a differential equation in which the dependent variable and its derivative occur only in <u>first degree</u> and not multiplied together. i.e, the differential equations which do not contains products, powers or ~~coeff~~ quotients of unknown functions and its derivatives.

Intial Value Problem.

An ODE, together with an initial condition is called an initial value problem. If the ODE is explicit (directly we can read), $y' = f(x, y)$, the initial value problem is of the form,

$$y' = f(x,y), \quad y(x_0) = y_0.$$

complimentary
singular particular
solution

21/6/16 General Form of an n^{th} order, linear, ordinary differential equation. The

The general form of an n^{th} order, linear ODE is of the following form

$$\boxed{a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \cdots\cdots + a_n y = g(x), \quad a_0 \neq 0}$$

- The D.E is said to be homogeneous, if $g(x) = 0$ and non-homogeneous, if $g(x)$ ~~is~~ not identically zero. $(g(x) \neq 0)$

[illegible] by [illegible] operator

Denoting $\dfrac{d^n y}{dx^n} = D^n y$

ie, `D' stands for the operator $\dfrac{d}{dx}$.

$$\text{ie, } \frac{dy}{dx} = \frac{d}{dx}(y) = \underline{Dy}.$$

$$\text{Similarly, } \frac{d^2y}{dx^2} = \frac{d^2(y)}{dx^2} = \frac{d}{dx}\cdot\frac{d}{dx}(y) = D \cdot D(y) = \underline{\underline{D^2 y}}.$$

Then the given equation can be rearranged in the form

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_n y = g(x)$$

$$[D^n + D^{n-1} \cancel{y} + D^{n-2} \cdots + a_n]\, y = g(x) \rightarrow \text{Symbolic form (SF)}$$

polynomial in D F(D)

$$F(D)\, y = g(x) \quad \text{—} ②$$

Solution

The solution of D.E is a function, $y(x)$ that satisfies the D.E

The general solution of ②, is of the form

(G.S)
y = The complementary function (CF) + the Particular ~~intes~~ integral (P.I)
(non-homogeneous)

Homogeneous ⇒ $\overset{(C.S)}{y} = C \cdot F$

Note:
The following the types of solution.

① General solution.
The solution containing arbitary constants, in which the number of arbitary constants is equal to the order of the differential equation is called the general solution or complete solution of the ~~di~~ D.E.

② Particular solution.
This is a solution obtained ~~by~~ from the general solution by assigning particular values to the arbitary constants.

③ Singular Solutions.
Any other solution is called singular solution (i.e, the solution does not contain any arbitary constants ~~is~~ called singular solutions)

Note:
The given D.E. may or may not have a singular solution. Since, $D = \frac{d}{dx}$ stands for the differentiation of x and $D^{-1} = \frac{1}{D}$ stands for integration with respect to x.

Homogeneous Linear Ordinary Differential Equation

In the above equation ~~do~~⑤, if $g(x)=0$ then the new equation $F(D)\,y = 0$ is called the homogeneous linear Ordinary differential equation.

- Higher Order Homogeneous ~~Linear~~ LODE's with constant coefficients form, $\frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots + a_n y = 0$

Steps for solving;

① Write the symbolic form of the given equation. i.e by, i.e, by replacing the differentials by operators.

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y \dots + a_n y = 0.$$

$[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n] y = 0$ → is called the symbolic form (S.F)

② Write the auxillary equation and solve it.

The auxillary equation can be found by replacing D by λ in the above equation.

i.e, $\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_n = 0$ → is called the Auxillary Equation (A.E)

Now we are going to solve the A.E for λ. and depending upon the nature of the roots we can write the complementary function (C.F)

③ Write solution as following

a) Since, the A.E is a polynomial of degree 'n' in λ, it will have 'n' roots say $\lambda_1, \lambda_2, \lambda_3 \dots \lambda_n$

Case 1

If the roots $\lambda_1, \lambda_2 \dots \lambda_n$ are real and distinct. then the C.F is i.e, y_c, i.e

i.e, $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + c_3 e^{\lambda_3 x} + \cdots\cdots + c_n e^{\lambda_n x}$

Case 2

If all the roots are real, but two of them are equal say $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3, \lambda_4 \cdots \lambda_n$

Then the C.F., i.e y_c, i.e $y = (c_1 + c_2 x)e^{\lambda x} + c_3 e^{\lambda_3 x} + \cdots$
$\cdots + c_n e^{\lambda_n x}$

Case 3

If all the roots are real, but three of them are equal say $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, $\lambda_4, \lambda_5 \cdots \lambda_n$

Then the CF, i.e, y_c i.e,

$$y = (c_1 + c_2 x + c_3 x^2)e^{\lambda x} + c_4 e^{\lambda_4 x} + \cdots + c_n e^{\lambda_n x}$$

Case 4

If the ~~A~~.E, ~~has~~ are complex roots say $\alpha \pm i\beta$, non repeated. Then the C.F, i.e, y_c i.e

$$y = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$$

Case 5

If the complex roots $\alpha \pm i\beta$, is repeated two time, then the C.F. i.e, y_c i.e,

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos\beta x + (c_3 + c_4 x)\sin\beta x]$$

Case 6.

If the complex roots repeating thricely.

C.F or y_c or $y = e^{\alpha x}[(c_1 + c_2 x + c_3 x^2)\cos\beta x + (c_4 + c_5 x + c_6 x^2)\sin\beta x]$

1) Solve the ODE $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$

ans: In symbolic form (S·F),

$$[D^3 - 2D^2 - D + 2]\,y = 0$$

then the Auxillary Equation (A E),

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \quad \Rightarrow \lambda^3 - 2\lambda^2 - \lambda + 2 = 0$$

$$* \quad \lambda = \frac{2 \pm \sqrt{2^2 - 4\times1\times1}}{2\times1}$$

$$= \frac{2 \pm \sqrt{4-4}}{2}$$

$$= \frac{2+}{2} \rightarrow \underline{\underline{\lambda = 1}}$$

$f(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$ a

$\lambda = 0$, $f(0) = 2 \neq 0$

$\lambda = 1$, $f(1) = 0$

$\therefore \lambda = 1$ is a root.

$(\lambda - 1)$ is a factor.

$$\begin{array}{r|rrrr} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

~~$f(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$~~
~~$= \lambda^2(\lambda - 2\lambda - \lambda + 2$~~

$$\lambda^2 - \lambda - 2 = 0$$

$x^2 - (a+b)x + ab$

$\boxed{\lambda^2 - \lambda - 2 = 0 \\ \lambda = 2, \lambda = }$

i.e

$$\lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 1)(\lambda^2 - \overset{\text{sum}}{\lambda} - \overset{\text{product}}{2}) \Rightarrow$$

$$\lambda^2 - \lambda - 2 \Rightarrow \lambda = 1, -1, 2$$

$\therefore$ The obtained 3 roots are real and distict i.e, $\lambda_1 = 1$, $\lambda_2 = -1$, $\lambda_3 = 2$.

$\boxed{\text{or}}$

$$x^2 - (a+b)x + ab = (x+a)(x+b)$$

$$\lambda = 1, -1, 2 \Rightarrow \lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = 2$$

The above obtained three roots are real and distinct.

$\therefore$ C.F $= y_c = y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + c_3 e^{\lambda_3 x}$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x}.$$

The complete solution (c.s), i.e

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x}.$$

2) $y^{(3)} + y' - 10y = 0.$

ans: $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} - 10y = 0$

the S.F, $D^3y + Dy - 10y = 0$

$[D^3 + D - 10]\, y = 0.$

the A.E, $\lambda^3 + \lambda - 10 = 0$

$f(\lambda) = \lambda^3 + \lambda - 10 = 0$

$\lambda = 0$, $f(0) = -10 \neq 0$

$\lambda = 1$, $f(1) = 1 + 1 - 10$
$= -8 \neq 0$

$\lambda = 2$, $f(2) = 2^3 + 2 - 10$
$= 8 + 2 - 10 = \underline{0}$

$\lambda = 2$ is a root, $(\lambda - 2)$ is a factor.

$$\begin{array}{c|cccc} 2 & 1 & 0 & 1 & -10 \\ & 0 & 2 & 4 & 10 \\ \hline & 1 & 2 & 5 & \boxed{0} \end{array}$$

$\lambda^2 + 2x + 5 = 0$

$ax^2+bx+c=0$

$\lambda^3 + \lambda - 10 = (\lambda - 2)(\lambda^2 + 2x + 5)$

$\lambda = 2$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 5}}{2 \times 1}$$

$$= \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4\sqrt{-1}}{2} \quad i = -1$$

$$= \frac{-2 \pm 4i}{2}$$

$$= \underline{\underline{-1 \pm 2i}}$$ complex ans.

The roots of the given A E become,

$\lambda = 2, \; -1 \pm 2i$

$\lambda_1 = 2, \; -1 \pm 2i$ ($\alpha \pm i\beta$ where $\alpha = -1, \beta = 2$)

∴ The C.F,

By case 4.

$y_c = \underset{\sim}{c_1} e^{\overset{-1}{\alpha} x} \underline{\underline{[c_2 \cos 2x + c_3 \sin 2x]}}$

$y_c = e^{\alpha x}[c_1 \cos \alpha x + c_2 \sin \beta x]$

Basics

3) Solve $9y^5 - 6y^4 + y^3 = 0$

ans:

The SF, $(9D^5 - 6D^4 + D^3)y = 0.$

The A.E, $9\lambda^5 - 6\lambda^4 + \lambda^3 = 0$

$\underline{\lambda^3}(9\lambda^2 - 6\lambda + 1) = 0$ $\lambda^3 = (0, 0, 0)$

$9\lambda^2 - 6\lambda + 1 = 0$ $ax^2 + bx + c = 0$

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4 \times 9 \times 1}}{2 \times 9}$$

$$= \frac{6 \pm \sqrt{36-36}}{18}$$

$=$

$$= \frac{6}{18} = \frac{3}{9} = \underline{\underline{\frac{1}{3}}} \text{ or } \underline{\underline{\frac{1}{3}}}$$

4)

H.W

5

Tut

$\lambda = 0, 0, 0 , \frac{1}{3} , \frac{1}{3}$

[illegible] repeat $\frac{1}{3}$ [illegible]

C. F or $y_c = y = (c_1 + c_2 x + c_3 x^2) e^{0x} + (c_4 + c_5 x) e^{1/3 x}$

i.e, The C.S or general solution

CF or $y_c = y = c_1 + c_2 x + c_3 x^2 + (c_4 + c_5 x) e^{1/3 x}$

4) The roots of the A.E's are

$\underline{3}, \underline{-5}, 0, 0, 0, 0, \underline{5}, 2 \pm 3i, 2 \pm 3i$

ans.

$2 \pm 3i \Rightarrow \alpha \pm \beta i$
$\alpha = 2$
$\beta = 3$

C.F =

$c_1 e^{3x} + c_2 e^{-5x} + c_3 e^{5x} + [c_4 + c_5 x + c_6 x^2 + c_7 x^3] e^{0x} +$

$e^{2x}[(c_8 + c_9 x)\cos 3x + (c_{10} + c_{11} x)\sin 3x$

H.W.

5) Solve ① $y''' + 9y' = 0$

① $\Rightarrow y = c_1 + c_2 \cos 3x + c_3 \sin 3x.$

Tutor ② $y^{IV} + 16y'' = 0$

② $\Rightarrow y = c_1 + c_2 x + c_3 \cos 4x + c_4 \sin 4x.$

③ $(D^4 + 10D^2 + 9I)y = 0$ $(I = 1)$

③ $\Rightarrow y = c_1 \cos x + c_2 \sin x + c_3 \cos 3x + c_4 \sin 3x.$

~~④ $y''' + 4 - 8 + y'$~~

6) $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$

$\Rightarrow$ root. 1, 3, 5

7) $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$

roots $\pm 1, \pm 2$

8) $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} + 12y = 0$

root, 1, 3, 4

Ans:
5) ① $y''' + 9y' = 0$

$$\frac{d^3y}{dx^3} + 9\frac{dy}{dx} = 0$$

S.F ⇒ $(D^3 + 9D).y = 0$

$\lambda = 0,$

$\lambda^2+9 =$

A.E ⇒ $\lambda^3 + 9\lambda = 0.$

$\lambda = -c \quad \lambda = \pm 3i$

$f(\lambda) = \lambda^3 + 9\lambda$

$\lambda = 0, \quad f(0) = \underline{\underline{0}}.$ $\quad\lambda = 0$ is root.

$$\begin{array}{c|cccc} 0 & 1 & 0 & 9 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline & 1 & 0 & 9 & \boxed{0} \end{array}$$

$\lambda^2 + 9 = 0$ $\qquad \lambda^3 + 9\lambda = (\lambda - 0)(\lambda^2 + 9)$

$\lambda^2 = -9 \Rightarrow \lambda = \sqrt{-9}$

$= 3\sqrt{-1} = \underline{\underline{\pm 3i}}.$

$\therefore \lambda = \underline{\underline{3i}}.$ (Here $\alpha = 0, \beta = 3$)

Then C.F or y_c or

$y = c_1 e^{0x} + c_2 \cos 3x + c_3 \sin 3x$

$y = c_1 + \underline{\underline{c_2 \cos 3x}} + c_3 \sin 3x$

② $y^{IV} + 16y'' = 0$

$$\frac{d^4y}{dx^4} + 16\frac{d^2y}{dx^2} = 0.$$

S.F ⇒ $(D^4 + 16D^2)y = 0$

A.E ⇒ $\lambda^4 + 16\lambda^2 = 0$

$\lambda^2(\lambda^2 + 16) = 0$

$\lambda^2 \Rightarrow \lambda = 0, 0$

$\lambda^2 + 16 = 0 \Rightarrow \lambda^2 = -16$

$\lambda = \sqrt{-16} = 4\sqrt{-1}$

$\therefore \lambda = \underline{\underline{\pm 4i}}. \; (\alpha = 0, \beta = 4)$

Then the CF is,

$y = (c_1 + c_2 x)e^{\lambda x} + c_3 \cos \beta x + c_4 \sin \beta x.$

$\therefore y = (c_1 + c_2 x)e^{\alpha x} + c_3 \cos 4x + c_4 \sin 4x$

$y = \underline{\underline{c_1 + c_2 x + c_3 \cos 4x + c_4}} \sin 4x.$

6) $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} \overset{-}{\cancel{+}} 15y = 0$

S.F → $(D^3 - 9D^2 + 23D - 15)y = 0$

A.E → $\lambda^3 - 9\lambda^2 + 23\lambda - 15 = 0$

$\lambda = 0, \quad f(0) = -15 \neq 0$

$\lambda = 1 \quad f(1) = 0.$, $\quad \lambda = 1$, is a root.

$$\begin{array}{r|rrrr} 1 & 1 & -9 & 23 & -15 \\ & 0 & 1 & -8 & 15 \\ \hline & 1 & -8 & 15 & \boxed{0} \end{array}$$

$\lambda^2 - 8\lambda + 15 = 0.$

$\lambda = \frac{8 \pm \sqrt{(-8)^2 - 4 \times 1 \times 15}}{2 \times 1} = \frac{8 \pm \sqrt{64-60}}{2}$

$= \frac{8 \pm \sqrt{4}}{2}$

$= \frac{8+2}{2}$ or $\frac{8-2}{2}$

roots

$= \underline{\underline{5}}$ or $\underline{\underline{3}}$

$\lambda = 1, 3, 5$

$CF = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}.$

7) $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 4y = 0$

S.F ⇒

$(D^4 - 5D^2 + 4)y = 0$

A E ⇒ $\lambda^4 - 5\lambda^2 + 4 = 0$

$\lambda = 0, \quad f(0) \neq 0$

$\lambda = 1, \quad f(1) = 0$

$\therefore \underline{\underline{\lambda = 1}}$ is a root

$$\begin{array}{r|rrrrr} 1 & 1 & 0 & -5 & 0 & 4 \\ & 0 & 1 & 1 & -4 & -4 \\ \hline & 1 & 1 & -4 & -4 & \boxed{0} \end{array}$$

$\lambda^3 + \lambda^2 - 4\lambda - 4 = 0$

$\lambda = 0, \quad f(0) \neq 0$

$\lambda = -1, \quad f(-1) = -1 + 1 + 4 - 4 = \underline{\underline{0}}$

$\underline{\underline{\lambda = -1}}$ is a root

$$\begin{array}{r|rrrr} -1 & 1 & 1 & -4 & -4 \\ & 0 & -1 & 0 & 4 \\ \hline & 1 & 0 & -4 & \boxed{0} \end{array}$$

$\lambda^2 - 4 = 0 \Rightarrow \lambda = \sqrt{4} \Rightarrow \lambda = \pm 2$

$\lambda = 2, \quad \lambda = -2$

$\underline{\underline{\lambda = 1, -1,}}\ 2, -2$

C.F ⇒

$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}.$

8) $\frac{d^3y}{dx^3} - 13\frac{dy}{dx} + 12y = 0$

SF⇒ $(D^3 - 13D + 12)y = 0$

A·E ⇒ ~~λ~~ $\lambda^3 - 13\lambda + 12 = 0$

$f(\lambda) = \lambda^3 - 13\lambda + 12$

$\lambda = 0, \quad f(0) \neq 0$

$\lambda = 1, \quad f(1) = 1 - 13 + 12 = -12 + 12 = \underline{\underline{0}}$

∴ $\underline{\underline{\lambda = 1}}$ is a root

$$\begin{array}{c|cccc} 1 & 1 & 0 & -13 & 12 \\ & 0 & 1 & 1 & -12 \\ \hline & 1 & 1 & -12 & \boxed{0} \end{array}$$

$\lambda^2 + \lambda - 12 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1 - 4 \times 1 \times -12}}{2} = \frac{-1 \pm \sqrt{1+48}}{2}$$

$$= \frac{-1 \pm \sqrt{49}}{2}$$

$$= \frac{-1 \pm 7}{2} = \frac{-1+7}{2} \text{ or } \frac{-1-7}{2}$$

$$= \frac{6}{2} \text{ or } \frac{-8}{2}$$

$$= \underline{\underline{3}} \text{ or } \underline{\underline{-4}}$$

roots, $\underline{\underline{\lambda = 1, 3, -4}}$

C·F is,

y_c or $y = c_1 e^x + c_2 e^{3x} + c_3 e^{-4x}$.

5) $(D^4$

22/1/16
HW Tutorial 1 9) $y^{IV} - 5y''' + 4y' = 0$

10) Solve the ODE, $y''' - 2y'' - y' + 2y = 0$

Ans) SFE ⇒ $$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} + 2y = 0$$

SF ⇒ $(D^3 - 2D^2 - D + 2)y = 0$

A·E ⇒ $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

$f(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2$

$\lambda = 0$, $f(0) = 2 \neq 0$

$\lambda = 1$ $f(1) = 1 - 2 - 1 + 2 = \underline{\underline{0}}$

$\lambda = 1$ is root.

$$\begin{array}{c|cccc} 1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$\lambda^2 - \lambda - 2 = 0$

$$\lambda = \frac{1 \pm \sqrt{1^2 - 4 \times 1 \times -2}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2} = \frac{1+3}{2} \text{ or } \frac{1-3}{2}$$

$= 2$ or $-\frac{2}{2}$

$= 2$ or -1

roots.

$\lambda = 1, 2, -1$.

The C.F or y_c or $y = c_1 e^{x} + \underline{\underline{c_2}} e^{2x} + c_3 e^{-x}$.

i.e, The complete solution (C.S) or general solution

$C.S = C.F$ (homogeneous)

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{-x}.$$

11) Solve, $y''' - y'' - 100y' - 100y = 0$. with the initial condition, $y(0) = 4$, $y'(0) = 11$, $y''(0) = -299$

ans: $$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} + 100\frac{dy}{dx} - 100y = 0$$

$$SF \rightarrow (D^3 - D^2 + 100D - 100)y = 0$$

$$A.E, \quad \lambda^3 - \lambda^2 + 100\lambda - 100 = 0$$

$(-1+1)$ [illegible]

$-1 = (-1)^2$

$\lambda = 0$, $f(0) = -100 \neq 0$

$\lambda = 1$ $f(1) = 1 - 1 + 100 - 100 ~~\neq 0~~ = 0$

$\lambda = (-1)$ $f(-1) =$ [illegible]

$\lambda = 1$ is root

$$\begin{array}{c|cccc} 1 & 1 & -1 & 100 & -100 \\ & 0 & 1 & 0 & 100 \\ \hline & 1 & 0 & 100 & \underline{\lfloor 0} \end{array}$$

$$\lambda^2 + 100 = 0$$

$$\lambda = \pm\sqrt{-100} = 10\sqrt{-1} = \underline{\underline{\pm 10i}}$$

$\alpha \pm i\beta$

$\alpha = 0$

$\beta = 10$

$$\lambda = 1, \pm 10i$$

$$C.F = c_1 e^{x} + c_2 e^{0x}[c_2 \cos 10x + c_3 \sin 10x]$$

or y_c or y = C.S or $y = C_1 e^{x} + C_2 \cos 10x + C_3 \sin 10x$.

- i.e, $y = c_1 e^{x} + c_2 \cos 10x + c_3 \sin 10x.$

- $y(0) = 4$

  put, $x = 0$

  $y(0) = c_1 e^{0x} + c_2 \cos 10 \times 0 + c_3 \sin 10 \times 0$ 12)

  $\sin 0 = 0$
  $\cos 0 = 1$

  $y = c_1 + c_2 \Rightarrow c_1 + c_2 = 4$ —— ① an.

- $y'(0) = 11$

- $y' = c_1 e^{x} - c_2 \sin 10x \times 10 + c_3 \cos 10x \times 10$

  $y' = c_1 e^{x} - 10c_2 \sin 10x + 10 c_3 \cos 10x.$

  $x = 0.$

  $y'(0) = c_1 + 10 c_3 \Rightarrow c_1 + 10c_3 = 11$ —— ②

- $y''(0) = -299$

- $y'' = c_1 e^{x} - 10 c_2 \cos 10x \times 10 - 10 c_3 \sin 10x \times 10$

  $y'' = c_1 e^{x} - 100 c_2 \cos 10x - 100 c_3 \sin 10x.$

  $x = 0$

  $y''(0) = c_1 - 100 c_2 \Rightarrow c_1 - 100 c_2 = -299$ —— ③

① - ③ $\Rightarrow$

$$c_1 + c_2 = 4$$
$$c_1 - 100c_2 = -299$$

~~$c_2 - 100c_2 = 4 - -299$~~ $101 c_2 = 303$

~~$c_2(1 - 100) = 303$~~

~~$c_2 = 303/99$~~ $\therefore c_2 = \frac{303}{101} = 3$

$$c_2 = 3$$

$c_1 + c_2 = 4$

$\therefore c_1 = 4 - c_2 = 4 - 3 = 1 \Rightarrow c_1 = 1$

Particular soulution, y_p i.e.

$$y = e^{x} + \underline{3\cos 10x} + \sin 10x$$

12) Solve the initial value problem.

$$y^{(3)} + 3y'' - 10y' = 0, \quad y(0) = 7, \; y'(0) = 0, \; y''(0) = 70$$

ans: S.F $\Rightarrow (D^3 + 3D^2 - 10D)y = 0$

A.E $\Rightarrow \lambda^3 + 3\lambda^2 - 10\lambda = 0$

$\lambda(\lambda^2 - 3\lambda - 10) = 0$

$\lambda = 0, \quad f(0) = \underline{\underline{0}}.$

$$\begin{array}{r|rrrr} 0 & 1 & 3 & -10 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline & 1 & 3 & -10 & \boxed{0} \end{array}$$

$$\lambda^2 + 3\lambda - 10 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times -10}}{2 \times 1}$$

$$= \frac{-3 \pm \sqrt{9 + 40}}{2} = \frac{-3 \pm 7}{2}$$

$$= \frac{-3+7}{2} \text{ or } \frac{-3-7}{2}$$

$$= 2 \text{ or } -5$$

$$\lambda = 0, \; -5, \; 2.$$

∴ The C.F or y_c or y

$$= c_1 e^{0x} + c_2 e^{-5x} + c_3 e^{2x}$$

c.s-r.s

$$y = c_1 + c_2 e^{-5x} + c_3 e^{2x}.$$

$y = c_1 + c_2 e^{-5x} + c_3 e^{2x}$

$y(0) = 7$

$x = 0$

$y(0) = c_1 + c_2 + c_3 \Rightarrow c_1 + c_2 + c_3 = 7 \longrightarrow ①$

T.W

tutorial 13

$y' = -5 c_2 e^{-5x} + 2 c_3 e^{2x}$

$x = 0 \quad y'(0) = 0$

Tutorial 14

$\cancel{y'(0) = c_1 + \frac{c_2}{-5} + \frac{c_3}{2}} \Rightarrow \cancel{c_1 - \frac{1}{5} c_2 + \frac{1}{2} c_3 = 0}$

$y'(0) = -5 c_2 + 2 c_3 \Rightarrow -5 c_2 + 2 c_3 = 0. \longrightarrow ②$

$y'' = -5 c_2 e^{-5x} \times -5 + 2 c_3 e^{2x} \times 2$

$y'' = 25 c_2 e^{-5x} + 4 c_3 e^{2x}.$

$x = 0$

$y''(0) = 25 c_2 + 4 c_3 \Rightarrow 25 c_2 + 4 c_3 = 70. \longrightarrow ③$

$② \times 5 \Rightarrow -25 c_2 + 10 c_3 = 0 \longrightarrow ④$

$④ + ③$

$$\begin{array}{r} -25 c_2 + 10 c_3 = 0 \\ 25 c_2 + 4 c_3 = 70 \\ \hline 0 + 14 c_3 = 70 \end{array}$$

$\Rightarrow c_3 = \frac{70}{14} = 5$

$\underline{c_3 = 5}$

$25 c_2 + 4 c_3 = 70$

$25 c_2 = 70 - 4 \times 5$

$c_2 = \frac{70 - 20}{25} = \underline{2} \Rightarrow \underline{c_2 = 2}$

$c_1 + c_2 + c_3 = 7 \Rightarrow c_1 = 7 - c_3 - c_2$

$= 7 - 5 - 2 \Rightarrow c = 0$

The particular solution, y_p ie,

$$y = 2e^{-5x} + 5e^{2x}.$$

H.W Tutorial

13) Solve Initial Value problem, $y''' + 3y'' + 4.81y' = 0$

$y(0) = 6,\ y'(0) = -3.15,\ y''(0) = -12.195$

Tutorial 1 $y_p = \frac{3}{2} + \frac{9}{4} e^{-3/2 x} \sin \frac{8x}{5} + \frac{9}{2} e^{-3/2} \cos 8x$

14) Solve the initial value problem,

$4y''' + 8y'' + 41y' + 37y = 0,$

$y(0) = 9,\ y'(0) = -6.5,\ y''(0) = -39.75,$
~~$y'''(0)$~~

ans: $y = 4e^{-x} + 5e^{-x/2} \cos 3x$

Second order homogeneous linear Ordinary Differential equation.

The standard form of the second order homogeneous linear ordinary differential equation is

$$y'' + p(x)y' + q(x)y = 0$$

Fundamental theorem of homogeneous linear differential equation of second order.

- For a homogeneous LODE,

$y'' + c(x)y' + q(x)y = 0$, any linear combination of two solutions on an open interval 'I' is again a solution of that equation.
In particular, for such an equation the sums and constant multiples of solutions are again solutions.

Note:
Thus if, $y = c_1 y_1 + c_2 y_2$ is also a solution where y_1 and y_2 are solutions and c_1, c_2 are

arbitary constants.

Initial value problem.(second order)

The initial value problem of second order is of the following form, $y'' + p(x)y' + q(x)y = 0$, $y(x_0) = y_0$ and $y'(x_0) = y_1$, two initial ~~Tv~~ conditions for second order.

Note:

- Two functions y_1 and y_2 define on an interval : I) are said to be linearly dependent, if one is a constant multiple of other
  ie, two functions y_1 and y_2 are called linearly ans. dependent on 'I' if, $K_1 y_1 + K_2 y_2 = 0$, wholes for some K_1 and K_2 not both zero.
- Otherwise, they are said to be linearly independent. ie, the two functions y_1 and y_2 are called linearly independent on an interval 'I' if $K_1 y_1 + K_2 y_2 = 0$, provided K_1 and K are both zero
- Then if, $K_1 \neq 0$, and $K_2 \neq 0$, we can see that y_1 and y_2 are proportional.. Consider $K_1 y_1 + K_2 y_2 = 0$, when $y_1 = \frac{-K_2 y_2}{K_1}$ and $y_2 = \frac{-K_1}{K_2} y_1$ $(K_1 \neq 0, K_2 \neq 0)$

General Solution, Basis Particular Solution.

A general solution of an ODE, $y'' + c(x)y' + q(x)y = 0$, on an open interval I, ie, $y = c_1 y_1 + c_2 y_2$ in which y_1 and y_2 are the solutions of the above equation on 'I'. That are not proportional, c_1 and c_2

are arbitary constants. These y_1, y_2 are called a <u>Basis</u>. (or Fundamental theorem) of the solution of the above homogeneous linear second order equation. A particular solution of the above equation on 'I', is obtained by assigning specific values c_1 and c_2 in,

$y = c_1 y_1 + c_2 y_2$ (General solution)

1) Solve the Initial value problem, $y'' + y = 0$, where $y(0) = 3$, $y'(0) = -0.5$.

ans: The S.F → $(D^2 + 1)y = 0$

A.E → $\lambda^2 + 1 = 0$. ~~put $\lambda = 0$, $f(0) = 0$~~

$$\cancel{\begin{array}{r|cccc} 0 & 1 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 0 \\ \hline & 1 & 0 & 1 & \boxed{0} \end{array}}$$

$\lambda^2 + 1 = 0 \Rightarrow \lambda^2 = -1$

$\lambda = \pm i$ $\qquad \alpha \pm i\beta.$ $\quad \alpha = 0, \beta = 1$

$\lambda = 0, \pm i$

C.F is

y_c ~~or~~ $y = \cancel{c_1 e^{ax}} + e^{0x}(c_1 \cos x + c_2 \sin x)$

C.S $= c_1 \cos x + c_2 \sin x$.

$y = c_1 \cos x + c_2 \sin x$,

$x = 0$

$y(0) = c_1$

$y(0) = 3$,

$\therefore c_1 = 3$

$y' = -c_1 \sin x + c_2 \cos x$.

$y'(0) = c_2$.

$y'(0) = -0.5$.

$\therefore c_2 = -0.5$

$\therefore$ The particular solution,

y_p ie $y = \underline{\underline{3\cos x - 0.5 \sin x}}$.

23/1/16

② Varify that, $y_1 = e^x$, $y_2 = e^{-x}$ are the solutions
* the ODE, $y'' - y = 0$, ~~then~~ (Hence) solve the initial
value problem $y'' - y = 0$, provided $y(0) = 6$, $y'(0) = $

$y_1'' - y_1 = 0, e^x$

$y_2'' - y_2 = 0$

ans:- Let, $y_1 = e^x$, $y_2 = e^{-x}$

$y_1' = e^x$ $\qquad$ $y_2' = -e^{-x}$ $\qquad$ $y_1 = e^x \to$

$y_1'' = e^x$ $\qquad$ $y_2'' = e^{-x}$ $\qquad$ $y_2'' - y_2 = 0$

$y_1'' - y_1 =$
$\underline{\underline{e^x - e^x = 0}}$

$y_2'' - y_2 = e^{-x} - e^{-x}$
$= \underline{\underline{0}}$

$\therefore$ $y_1 = e^x$ and $y_2 = e^{-x}$ are the solution of the given ODE, $y'' - y = 0$ —①

~~Then~~ (Fundamental theorem)

Now, $\frac{y_1}{y_2} = \frac{e^x}{e^{-x}} = e^{-2x} \neq$ constant.

$\therefore$ y_1 and y_2 are linearly independent, then from the fundamental theorem the General Solution (G.S) of the the given differential equation is, $y = c_1 y_1 + c_2 y_2$

ie, $y = c_1 e^x + c_2 e^{-x}$ —②

$y' = c_1 e^x - c_2 e^{-x}$

$y'' = c_1 e^x + c_2 e^{-x}$ —③

substitute ③ and ② in e①

$y'' - y' = 0 \Rightarrow c_1 e^x + c_2 e^{-x} \underline{\underline{- c_1 e^x - c_2 e^{-x}}} = 0$ in

$y = C_1 e^{x} + C_2 e^{-x}$

$y(0) = 6$

$y(0) = C_1 + C_2$

$C_1 + C_2 = 6$ —— ①

$y' = C_1 e^{x} - C_2 e^{-x}$

$y'(0) = -2$

$y'(0) = C_1 - C_2$

$C_1 - C_2 = -2$ —— ②

① + ② $\Rightarrow$

$$\begin{array}{rl} C_1 + C_2 & = 6 \\ C_1 - C_2 & = -2 \\ \hline 2C_1 & = 4 \end{array}$$

$\therefore \underline{\underline{C_1 = 2}}$

$C_1 = \frac{4}{2} = \underline{\underline{2}}$

$C_1 + C_2 = 6 \Rightarrow C_2 = 6 - C_1$

$= 6 - 2$

$= \underline{\underline{4}}$

$\underline{\underline{C_2 = 4}}$

Then Particular solution, y_p i.e

$$\underline{\underline{y = 2e^{x} + 4e^{-x}}}$$

Find a Basis if one solution is known (Reduction of Order)

If a non-zero solution, y_1 of the homogeneous ODE is known; Then the second linearly independent solution y_2 can be obtained by as follows,

$y_2 = V y_1$, where $\boxed{V = \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx}$.

① Find a Basis of the solutions of the Ordinary D.E. $(x^2 - x)y'' - xy' + y = 0$

<u>imp</u>

$\boxed{y_1 = x}$

$y' = 1$

By inspection;

$y_1 = x$, $y_1' = 1$, $y_1'' = 0$.

$(x^2 - x)y_1'' - xy' + y = 0$.

$(x^2 - x) \times 0 - x \times 1 + x = 0$ (basis)

$\therefore y_1 = x$ is a solution of the given D.E

~~By inspection,~~

Now, we are going to find y (second basis)

$y_2 = V \cdot y_1$

$$V = \int \frac{e^{\int p\,dx}}{y_1^2}\,dx$$

~~For that we are comparing the given equating with standard equat~~

~~$(x^2 - x)y'' - xy' + y = 0$~~

~~$y'' - p(x)y' + q(x)y = 0$~~

~~comparing the above 2 equations, we get~~

~~$p = -x$~~

~~Then~~, Rearraning the given equating by deviding through $x^2 - x$,

$$\frac{x}{x^2-x} = \frac{x}{x(x-}$$

$(x^2 - x)y'' - xy' + y = 0$

$$y'' - \frac{x}{x^2-x}y' + \frac{y}{x^2-x} = 0$$

$$y'' - \frac{1}{x-1}y' + \frac{y}{x^2-x} = 0 \quad \text{①}$$

comparing this equating ① with the standard equat

$y'' + - p(x)y' + q(x)y = 0$

we get $p = -\frac{1}{x-1}$

$$\int p\,dx = \int -\frac{1}{x-1}\,dx.$$

$$= -\int \frac{1}{x-1}\,dx$$

$$= -\log(x-1).$$

$e^{-\int p\,dx} = e^{\log(x-1)}$

$= \underline{\underline{x - 1}}$

$e^{\log x} = x$.

$x - 1$ //

$v = \int \frac{x-1}{x^2}\,dx$

$= \int \frac{x}{x^2} - \frac{1}{x^2}\,dx$

$= \int \frac{1}{x} - \frac{1}{x^2}\,dx.$

$\frac{x^{-2+1}}{-2+1} =$

$= \int \frac{1}{x}dx - \int x^{-2}\,dx$

$= \log|x| - \left(\frac{x^{-1}}{\cancel{x}-1}\right)$

$= \underline{\underline{\log|x| + \frac{1}{x} + c}}$

∴ the second basis $y_2 = vy_1 = \left(\log x + \frac{1}{x}\right) \cdot x$

~~$= x \log x +$~~

$= x \log x + \frac{x}{x}$

$y_2 = \underline{\underline{x \log x + 1}}$

∴ the basis of the given LDE are,

$\underline{y_1 = x \;\&\; y_2 = x \log x + 1}$

② Solve the following D.E, by the reduction of order ~~&~~ $y'' - y' = 0$.

$y'' - y' = 0.$

$y_1 = e^x, \quad y_1' = e^x, \quad y_1'' = e^x$

ie, $y_1'' - y' = e^x - e^x = 0.$

$y_2 = vy_1$

Comparing the given equations $y'' - y' = 0$ with the standard equation ~~$y'' + p(x)y' +$~~

$y'' + p(x)y' = q(x)y = 0$

$P = -1$

$\int p\,dx = \int -\,dx = \underline{\underline{-x}}$

$e^{\int -p\,dx} = e^{x}$

$V = \int \frac{\cancel{e^{x}}}{e^{x} \times \cancel{e^{x}}} = \int \frac{1}{e^{x}}\,dx$

$= \int e^{-x}\,dx = -e^{-x}$

$y_2 = V y_1$

$y_2 = -e^{-x} \times e^{x} = -e^{x-x} = -e^{0} = \underline{\underline{-1}}$

∴ The basis of the given ODE are

$\underline{\underline{y_1 = e^{x}}}, \quad y_2 = \underline{\underline{\cancel{e} -1}}$

Second Order Homogeneou Linear Ordinary Differential Equation with constant coefficients.

The ~~Second~~ standard form of the second order ordinary linear differential equation with constant coefficient is $y'' + ay' + by = 0$ where a and b are arbitary constants

① Solve the ODE $y'' - y = 0$

ans: the SF is ⇒

$(D^2 - 1)y = 0$

A·E ⇒ $\lambda^2 - 1 = 0$ $\quad \lambda = \underline{\underline{\pm 1}}$, ie, $\lambda = 1, -1$

The CF ie y_c ie

$$y = c_1 e^{x} + c_2 e^{-x}$$

G.S, $y = c_1 e^{x} + c_2 e^{-x}$

② Solve the ODE, $y'' - 0.25 y = 0$

ans: SF ⇒ $(D^2 - 0.25) y = 0$

A.E ⇒ $\lambda^2 - 0.25 = 0$

$$\lambda^2 = 0.25 \qquad \lambda = \sqrt{0.25} \Rightarrow \lambda = \pm 0.5$$

$\lambda = 0.5, -0.5$ is roots.

The C.F is, y_c ie,

$$y = c_1 e^{0.5x} + c_2 e^{-0.5x}$$

The G.S is,

$$y = c_1 e^{0.5x} + c_2 e^{-0.5x}$$

Tutorial 2

③ Solve the ODE, ⓐ $y'' + 6y' + 9y = 0$

ⓑ $9y'' - 30y' + 25y = 0$

ⓒ $4y'' - 4y' - 3y = 0$.

④ $y'' + 1.75 y' - 0.5 y = 0$

ans: S.F ⇒ $(D^2 + 1.75 D - 0.5) y = 0$

AE ⇒ $\lambda^2 + 1.75\lambda - 0.5 = 0$

$$\lambda = \frac{-1.75 \pm \sqrt{(1.75)^2 - 4 \times 1 \times -0.5}}{2 \times 1}$$

$$= \frac{-1.75 \pm \sqrt{5.0625}}{2} = \frac{-1.75 + \sqrt{5.0625}}{2} \quad \text{or} \quad \frac{-1.75 - \sqrt{5.062}}{2}$$

$$= 8.0375$$

$\frac{-1.75+2.25}{2}$ or $\frac{-1.75-2.25}{2}$

~~$= \frac{\sqrt{8.375}}{2}$ or~~

$= \frac{.5}{2}$ or $\frac{-4}{2}$

$= 0.25$ or -2

$= \underline{\underline{\frac{1}{4}}}$ or $\underline{\underline{-2}}$

C.F is, ie, y_c .ie $y = c_1 e^{1/4 x} + c_2 e^{-2x}$

G.S is $y = \underline{\underline{C_1 e^{1/4 x} + c_2 e^{-2x}}}$

⑤ Q. $y'' + 2\pi y' + \pi^2 y = 0$, Initial values $y(0) = 4.5$, $y'(0) = -4.5\pi$
$= 13.137$

S.F $\Rightarrow (D^2 + 2\pi D + \pi^2) y = 0$

A.E $\Rightarrow \lambda^2 + 2\pi \lambda + \pi^2 = 0$

$\lambda = \frac{-2\pi \pm \sqrt{(2\pi^2) - 4 \times 1 \times \pi^2}}{2}$

$= \frac{-2\pi \pm \sqrt{4\pi^2 - 4\pi^2}}{2}$

$= \frac{-2\pi}{2} = \underline{\underline{-\pi}}$

(λ repeat 2, becau quadratic have 2 λ's in case const -π.)

$\lambda = \underline{\underline{-\pi}}, -\pi$ is root

~~C.F, $y = c_1 e^{-\pi x} + c_2 e^{-\pi x}$~~

G.S is $y = (C_1 + C_2 x) e^{-\pi x}$

ans > ~~$y = c_1 x + c_2 x e^{-}$~~ $y = c_1 e^{-\pi x} + x \; c_2 e^{\cdot\cdot}$

$y = y(x)$

$x = 0$
$y(0) = c_1 \Rightarrow \underline{\underline{c_1 = 4.5}}$ — ①

$y' = -c_1 e^{\pi x} \times \pi \left(x \times - c_2 e^{-\pi x} \times \pi + c_2 e^{-\pi x} \times 1 \right)$ exercise.

~~$y' = -c_1 e^{-x} - x c_2 e^{-\pi x} + c_2 e^{-\pi x}$~~

$y' = -\pi c_1 e^{-\pi x} - \pi x c_2 e^{-\pi x} + c_2 e^{-\pi x}$

$y'(0) = -\pi c_1 + c_2 \Rightarrow -\pi c_1 + c_2 = 13.137$ ~~②~~

$\Rightarrow c_2 - \pi c_1 = 13.137$ — ②

~~$c_2 = \pi c_1$~~ $c_2 - \pi c_1 = 13.137$,

= ~~3.14×4.5~~ ~~$c_2 = 13.137 + \pi c_1$~~

~~$c_2 = 13.137 + \pi \times 4.5$~~

$c_2 - \pi c_1 = 13.137$

$c_2 - 4.5\pi = -4.5\pi - 1$

$c_2 = -4.5\pi + 4.5\pi - 1$

$\therefore \underline{\underline{c_2 = -1}}$

) Solve the Initial value problem

$y'' + y' - 2y = 0.$

where $y(0) = 4, \; y'(0) = -5$

⑦

ans: SF ⇒ $(D^2 + D - 2)y = 0$

AE ⇒ $\lambda^2 + \lambda - 2 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 - 2}}{2 \times 1}$$

$$= \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1+3}{2} \text{ or } \frac{-1-3}{2}$$

$$= \frac{2}{2} \text{ or } \frac{-4}{2}$$

$$= \underline{1} \text{ or } \underline{\underline{-2}}$$

roots are

$\lambda = 1, -2.$

C.F ie, y_c ie $y = c_1 e^{x} + c_2 e^{-2x}$

~~$y = c_1$~~ G.S $= c_1 ex + c_2 e^{-2x}$

$y(x) = c_1 e^{x} + c_2 . e^{-2x}$

$y(0) = 4.$

$y(0) = c_1 + c_2$

$\Rightarrow c_1 + c_2 = 4$

$y'(0) = -5$

$y'(x) = c_1 e^{x} - 2c_2 e^{-2x}$

$y'(0) = c_1 - 2c_2$

$\Rightarrow c_1 - 2c_2 = -5$

① - ② ⇒

$$\begin{array}{r} c_1 + c_2 = 4 \\ c_1 - 2c_2 = -5 \\ \hline 3c_2 = 9 \end{array}$$

⇒ $\underline{\underline{c_2 = 9/3}}$

$c_1 + c_2 = 4$ ⇒ ~~$c_2 = 4 - c_1$~~

$c_1 = 4 - c_2$ ~~$c_1 = 4 - 9$~~ $c_1 = 1$

$c_1 = 4 - 9/3 = \frac{12-9}{3} = \frac{3}{3} = \underline{\underline{1}}$ $\underline{\underline{c_2 = \frac{9}{3} = 3}}$

$y_p = e^{x} + 3e^{-2x}$ //

⑦ $y'' + 2y' + y = 0$, $y(0) = 5$, $y'(0) = -3$

ans. SF $\Rightarrow$ $(D^2 + 2D + 1)y = 0$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 1}}{2}$$

$$= \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2}{2} = -1$$

roots $-1, -1$

$\lambda =$ $=$ repat -2

c.F y_1 ie ~~$y = c_1 e^{-x} + c_2 e^{-x}$.~~

$$y = (c_1 + c_2 x)e^{-x}$$

~~$y(0) = 5$~~

~~$y = c_1 e^{-x} + c_2 e^{-x}$.~~

~~$y(0) = c_1 + c_2$~~

~~$c_1 + c_2 = 5$ — ①~~

① − 2

~~$y' = -c_1 e^{-x} - c_2 e^{-x}$~~

~~$y'(0) = -3$~~

~~$y'(0) = -c_1 - c_2$~~

~~$-c_1 - c_2 = -3$ ②~~

$y(0) = 5$

$y(0) = c_1 e^{-0} = \underline{\underline{c_1}}$. $\underline{\underline{c_1 = 5}}$

$y'(0) = -3$

$c_2 = 2$

$y(x) = c_1 e^{-x} +$ $x\, c_2 e^{-x}$.

$y'(x) = -c_1 e^{-x} + (x \times -c_2 e^{-x} + c_2 e^{-x} \times 1)$

$y'(x) = -c_1 e^{-x} - x c_2 e^{-x} + c_2 e^{-x}$

$y'(0) = -c_1 + c_2$ $\Rightarrow$ $c_2 - c_1 = -3$ — ②

substitue c_1 to ②.

$c_2 - c_1 = -3$

$c_2 - 5 = -3$

$c_2 = -3 + 5 = 2$

$\underline{\underline{c_2 = 2}}$

particular solution, y_p ie

$y = 5e^{-x} \underline{+ 2x} e^{-x}$

⑧ HW Q $y^4 - 9y^2 - 400y = 0$

$y(0) = 0, \; y'(0) = 0, \; y''(0) = 41, \; y'''(0) = 0$

$(\text{ans}: \cosh 5x - \cos 4x)$

~~Homogeneous~~

Higher Order Homogeneous LODE.

Continue.

5) ③. $(D^4 + 10D^2 + 9I)y = 0$

SF ⇒ $(D^4 + 10D^2 + 9)y = 0$ $\quad I = 1$

AE ⇒ $\lambda^4 + 10\lambda^2 + 9 = 0$

put $\lambda^2 = k$

⇒ ∴ $k^2 + 10k + 9 = 0$

$$k = \frac{-10 \pm \sqrt{10^2 - 4 \times 1 \times 9}}{2 \times 1}$$

$$= \frac{-10 \pm \sqrt{64}}{2} = \frac{-10 \pm 8}{2}$$

$$= \frac{-10+8}{2} \text{ or } \frac{-10-8}{2}$$

~~The roots are~~ $= \underline{\underline{-1}}$ or $\underline{\underline{-9}}$

$\lambda^2 = -1 \Rightarrow \lambda = \pm i$ $(\alpha = 0, \beta = 1)$

$\lambda^2 = -9 \Rightarrow \lambda = \sqrt{-9} = \pm 3i$

∴ The roots are, $\pm i$ and $\pm 3i$ $(\alpha = 0, \beta =$

The CF ie y_c ie

$y = e^{0x}[c_1 \cos x + c_2 \sin x] + e^{0x}[c_3 \cos 3x + c_4 \sin 3x]$

∴ $y = c_1 \cos x + c_2 \sin x + \underline{c_3 \cos 3x} + c_4 \sin 3x$

9) $y^{iv} - 5y''' + 4y' = 0$

SF ⇒ $(D^4 - 5D^3 + 4D)y = 0$

AE ⇒ $\lambda^4 - 5\lambda^3 + 4\lambda = 0$

$\lambda(\lambda^3 - 5\lambda^2 + 4) = 0$

$\lambda = 0$ is a root

$f(\lambda) = \lambda^3 - 5\lambda^2 + 4$

$\lambda = 0$, $f(0) = 4 \neq 0$

$\lambda = 1$ $\quad f(1) = 1 - 5 + 4 = \underline{\underline{0}}$

$\lambda = 1$, is a root

$$\begin{array}{c|cccc} 1 & 1 & -5 & 0 & 4 \\ & 0 & 1 & -4 & -4 \\ \hline & 1 & -4 & -4 & \boxed{0} \end{array}$$

$\lambda^2 - 4\lambda - 4 = 0$

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4\times1\times-4}}{2\times1}$$

$$\lambda = \frac{4 \pm \sqrt{16+16}}{2} = \frac{4 \pm \sqrt{32}}{2}$$

$$= \frac{4 \pm 4\sqrt{2}}{2}$$

The roots are $= 2 \pm 2\sqrt{2}$ ⇒ roots, $\underline{\underline{2+2\sqrt{2}}}$ or $\underline{\underline{2-2\sqrt{2}}}$

$\lambda = 0, 1, 2+2\sqrt{2}, 2-2\sqrt{2}$

The CF i'e, y_c i'e

$y = c_1 e^{0x} + c_2 e^{x} + c_3 e^{(2+2\sqrt{2})x} + c_4 e^{(2-2\sqrt{2})x}$

The G.S ie

$y = c_1 + c_2 e^{x} + c_3 e^{(2+2\sqrt{2})x} + c_4 e^{(2-2\sqrt{2})x}$

Tutorial 2 (ans:)

Second Order Homogeneous LODE's (continues)

③ ⓐ $y'' + 6y' + 9y = 0$

(ans SF $\Rightarrow (D^2 + 6D + 9)y = 0$

HW

AE $\Rightarrow \lambda^2 + 6\lambda + 9 = 0$

$$\lambda = \frac{-6 \pm \sqrt{6^2 - 4 \times 1 \times 9}}{2}$$

$$= \frac{-6 \pm \sqrt{36 - 36}}{2} = \frac{-6}{2} \ \& \ -\frac{6}{2}$$

$$= \underline{\underline{-3}} \ \& \ \underline{\underline{-3}}$$

C.F ie y_c ie $y = (c_1 + c_2 x) e^{-3x}$.

G.S ie $\underline{\underline{y = (C_1 + C_2 x) e^{-3x}}}$

ⓑ $9y'' - 30y' + 25y = 0$

ans. SF $\Rightarrow (9D^2 - 30D + 25) y = 0$

AE $\Rightarrow 9\lambda^2 - 30\lambda + 25 = 0$

$$\lambda = \frac{30 \pm \sqrt{(30)^2 - 4 \times 9 \times 25}}{2 \times 9}$$

$$= \frac{30 \pm \sqrt{900 - 900}}{18}$$

$$= \frac{30 \pm}{18} = \underline{\underline{\frac{5}{3}}} \ \& \ \underline{\underline{\frac{5}{3}}}$$

roots

$\lambda = \frac{5}{3}, \frac{5}{3}$

CF ie y_c ie $y = (c_1 + \underline{\underline{c_2 x}}) e^{5/3 x}$

GS ie $y = (C_1 + C_2 x) e^{5/3 x}$.

5×3
$\frac{15}{30}$
18
$9 \ 3 \times 3$

$\frac{15}{9}$.

ⓒ $4y'' - 4y' - 3y = 0$

ans: SF ⇒ $(4D^2 - 4D - 3)y = 0$

AE ⇒ $4\lambda^2 - 4\lambda - 3 = 0$

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 4 \times -3}}{2 \times 4}$$

$$= \frac{4 \pm \sqrt{16 + 48}}{8}$$

$$= \frac{4 \pm \sqrt{64}}{8} = \frac{4 \pm 8}{8} = \frac{4+8}{8} \;\&\; \frac{4-8}{8}$$

$$= \frac{12}{8} \;\&\; -\frac{4}{8}$$

$$= \frac{3}{2} \;\&\; -\frac{1}{2}$$

$$\frac{6 \times 2}{8\; 4 \times 2} = \frac{6\; 3 \times 2}{4\; 2 \times 2}$$

The roots are

$\lambda = \frac{3}{2}, \frac{-1}{2}.$

The CF i'e y_c i'e

$$y = c_1 e^{3/2 x} + c_2 e^{-1/2 x}$$

G.S i'e $y = \underline{\underline{c_1 e^{3/2 x}}} + c_2 e^{-1/2 x}$

Tutorial 2

② 9.) Reduction of order

Find the basis by the method of reduction of order.

HW

$x^2y'' + xy' - 4y = 0$

$y = x^2$

$p = \frac{1}{x}$

10) Solve the initial value problem

$y'' + y' + 0.25y = 0$

$y(0) = 3.0, \; y'(0) = -3.5$

ans:

$S F \Rightarrow (D^2 + D + 0.25)y = 0$

$A \cdot E \rightarrow \lambda^2 + \lambda + 0.25 = 0$

$$\lambda = \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 0.25}}{2}$$

$$= \frac{-1 \pm \sqrt{1-1}}{2} = \frac{-1 \pm}{2} = \underline{\underline{\frac{-1}{2}}}, \underline{\underline{\frac{-1}{2}}}$$

$$\lambda = \frac{-1}{2}, \frac{-1}{2}$$

The CF, is, y_c i.e

$$y = (c_1 + c_2 x) e^{-1/2 x}$$

$y(0) = 3.0$

$y = (c_1 + c_2 x) e^{-1/2 x}$

$x = 0$

$y(x) = c_1 e^{-1/2 \times 0} = \underline{\underline{c_1}}$

$\therefore c_1 = \underline{\underline{3}}$ —①

$y'(0) = -3.5$

$y'(x) =$

$= -\frac{1}{2} c_1 e^{-1/2 x}$

$y'(0) = -3.5$

$y = c_1 e^{-1/2 x} + c_2 (x e^{-1/2 x})$

$$y'(x) = \frac{-1}{2} c_1 e^{-1/2 x} + c_2 \left(x \cdot \frac{-1}{2} e^{-1/2 x} + e^{-1/2 x} \times 1\right)$$

$$y'(x) = \frac{-1}{2} c_1 e^{-1/2 x} - \frac{1}{2} c_2 x e^{-1/2 x} + c_2 e^{-1/2 x}$$

$x = 0$

$y'(0) = -\frac{1}{2} c_1 + c_2$

$y'(0) = -3.5$, $\quad \frac{-1}{2} c_1 + c_2 = -3.5$ —②

Substitute ① in ② $\Rightarrow$

$$-\frac{1}{2} \times 3 + C_2 = -3.5$$

$$-\frac{3}{2} + C_2 = -3.5$$

$$C_2 = -3.5 + \frac{3}{2}$$

$$= -\frac{3.5 \times 2 + 3}{2} = \frac{-7+3}{2} = \frac{-4}{2} = \underline{\underline{-2}}$$

$\therefore$ $\underline{C_1 = 3}$, $\underline{\underline{C_2 = -2}}$

$\therefore$ The particular solution, ie y_p ie

$$y = (3 \overset{-}{\cancel{+}} 2x) e^{-\frac{1}{2}x}$$

$$\therefore y = \underline{\underline{3 e^{-\frac{1}{2}x} - 2x e^{-\frac{1}{2}x}}}$$

11) Solve the initial value problem

$y'' + 0.4 y' + 9.04 y = 0$, $y(0) = 0$ & $y'(0) = 3$.

ans: SF $\Rightarrow$ $(D^2 + 0.4D + 9.04)y = 0$

AE $\Rightarrow$ $\lambda^2 + 0.4\lambda + 9.04 = 0$

$$\lambda = \frac{-0.4 \pm \sqrt{(0.4)^2 - 4 \times 1 \times 9.04}}{2 \times 1}$$

$$= \frac{-0.4 \pm 6i}{2} = \underline{\underline{-0.2 \pm 3i}} \; (\alpha = -0.2, \beta = 3)$$

C.F ie y_c ie $y = \underline{\underline{e^{-0.2x}[C_1 \cos 3x + C_2 \sin 3x]}}$

$\therefore$ The G.S ie

$$y = c_1 e^{-0.2x} \cos 3x + c_2 e^{-0.2x} \sin 3x.$$

$y(0) = 0$

$\Rightarrow$ $y(0) = c_1$ $\Rightarrow$ $\underline{\underline{C_1 = 0}}$ —①

$y = c_1 e^{-0.2x} \cos 3x + c_2 e^{-0.2x} \sin 3x$

$y'(x) = c_1 \big(e^{-0.2x} \times 3x - \sin 3x + \cos 3x \times {}^{-0.2x} e^{-0.2x} \big) +$

$c_2 \big(e^{-0.2x} \times 3x \cos 3x + \sin 3x \times -0.2x\, e^{-0.2x}$

$y'(x) = -3c_1 e^{-0.2x} \sin 3x - 0.2\, c_1 e^{-0.2x} \cos 3x +$

$3c_2 e^{-0.2x} \cos 3x - 0.2\, c_2 e^{-0.2x} \sin 3x$

$x = 0$

$y'(0) = -0.2\, c_1 + 3c_2$

$y'(0) = 3 \quad \Rightarrow \quad -0.2\, c_1 + 3c_2 = 3 \quad — ②$

by ①

$c_1 = 0$

$\therefore$ ② $\Rightarrow -0.2 \times 0 + 3c_2 = 3$

$\Rightarrow 3c_2 = 3$

$\Rightarrow \underline{\underline{c_2 = 1}}$

$\therefore$ The particular solution, ie y_p ie

$y = 0 \times e^{-0.2x} \cos 3x + 1 \times e^{-0.2x} \sin 3x$

$\therefore \underline{y = e^{-0.2x} \sin 3x}$

Reduction of order

12) Find an ODE, $y'' + ay' + by = 0$ for the basis $e^{-\sqrt{2}x}$ and $x \cdot e^{-\sqrt{2}x}$

ans: Given basis are $e^{-\sqrt{2}x}$ and $x \cdot e^{-\sqrt{2}x}$ then by the fundamental theorem, the General solution (G.S) ie,

$y = c_1 y_1 + c_2 y_2$

Here $y_1 = e^{-\sqrt{2}x}$, $y_2 = x e^{-\sqrt{2}x}$

$y = c_1 e^{-\sqrt{2}x} + c_2 x e^{-\sqrt{2} x}$

$\therefore y = (c_1 + c_2 x) e^{-\sqrt{2} x}$ —— ①

Differentiating ① with respect to x.

$y'(x) = -\sqrt{2}\, c_1 e^{-\sqrt{2} x} + c_2 (x \times -\sqrt{2} e^{-\sqrt{2}x} + e^{-\sqrt{2}x} \times 1)$

$y'(x) = -\sqrt{2}\, c_1 e^{-\sqrt{2} x} - \sqrt{2}\, c_2 x e^{-\sqrt{2} x} + c_2 e^{-\sqrt{2} x}.$

$y'(x) = \underline{(c_1 + c_2 x)} \times \sqrt{2}\, \underline{e^{-\sqrt{2} x}} + e^{-\sqrt{2} x} c_2.$

$y' = -\sqrt{2}\, y + c_2 e^{-\sqrt{2} x}.$

$y' + \sqrt{2}\, y - c_2 e^{-\sqrt{2} x} = 0$ —— ②

Differentiating the ② with respect to x.

$y'' + \sqrt{2}\, y' - c_2 \times -\sqrt{2}\, e^{-\sqrt{2} x} = 0$

$y'' + \sqrt{2}\, y' + \sqrt{2}\, \underbrace{c_2 e^{-\sqrt{2} x}} = 0$ —— ③

by ② $\Rightarrow$ $y' + \sqrt{2}\, y = c_2 e^{-\sqrt{2} x}$

substitute to ③ $\Rightarrow$

$y'' + \sqrt{2}\, y' + \sqrt{2}\,(y' + \sqrt{2}\, y) = 0$

$y'' + \sqrt{2}\, y' + \sqrt{2}\, y' + 2y = 0$

$\underline{\underline{y'' + 2\sqrt{2}\, y' + 2y = 0}}$ be the required ODE.

13) Find an ODE, for the basis $e^{-0.5x}$ & $e^{-2.5x}$

ans: Given basis are $e^{-0.5x}$ & $e^{-2.5x}$

~~$y = c_1 y_1 + c_2 y_2$~~

Let $y_1 = e^{-0.5x}$ and $y = e^{-2.5x}$ are the basis of the expected D.E

Then by fundamental theorem we hav the G.S ie, $y = c_1 y_1 + c_2 y_2$

ie, $y = c_1 e^{-0.5x} + c_2 e^{-2.5x}$ — ①

Differentiating, the ① with respect to x

ie, $y' = -0.5\, c_1 e^{-0.5x} - 2.5\, c_2 e^{-2.5x}$ — ②

Differentiating the ② with respect to x we get

$y'' = 0.5\times0.5\, c_1 e^{-0.5x} + 2.5\times2.5\, c_2 e^{-2.5x}$

$y'' = 0.25\, c_1 e^{-0.5x} + 6.25\, c_2 e^{-2.5x}$ (no arbita constan in DE

$y'' = 0.25\left(c_1 e^{-0.5x} + 25\, c_2 e^{-2.5x}\right)$ 25

Replacement by c_1 and c_2 by ①

$y = c_1 e^{-0.5x} + c_2 e^{-2.5x}$.

$c_1 e^{-0.5x} = y - c_2 e^{-2.5x}$

$\therefore y'' = 0.25\left(y - c_2 e^{-2.5x} + 25\, c_2 e^{-2.5x}\right)$

Again replacing ① d

$c_2 = \dfrac{y - c_1 e^{-0.5x}}{e^{-2.5x}}$

14) Find a second order linear ODE with given basis, $e^{-2.5x}\cos(.5x)$ and $e^{-2.5x}\sin(.5x)$ and the initial condetion $y(0) = 1.5$ and $y'(0) = -2.0$

ans. Here we are given the two basis of a expected second order Linear D-E

Then by the fundamental theorem.

ie, $y = C_1 y_1 + C_2 y_2$

$$y = C_1 e^{-2.5x} \cos 6.5x) + C_2 e^{-2.5x} \sin (0.5x.)$$

$\therefore y = e^{-2.5x}(c_1 \cos 0.5x + c_2 \sin 0.5x)$

$\alpha = -2.5, \beta = 0.5 \qquad -2.5 \pm 0.5i$

ie again

$y = y(x) = c_1 e^{-2.5x} \cos 0.5x + c_2 e^{-2.5x} \sin 0.5x$

$y(0) = 1.5$

$y(0) = C_1 \Rightarrow c_1 = 1.5$ ①

$y'(x) = c_1[(e^{-2.5x} \times 0.5x - \sin 0.5x + \cos(0.5x) + -2.5 \times e^{-2.}$

$+ c_2[e^{-2.5x} \times 0.5 \times \cos 0.5x + \sin 0.5x \times -2.5 e^{-2.}$

$y'(x) = -0.5 c_1 e^{-2.5x} \sin 0.5x - 2.5 c_1 e^{-2.5x} \cos(0.5x$

$+ 0.5 c_2 e^{-2.5x} \cos 0.5x - 2.5 c_2 e^{-2.5x} \sin 0.5$

$y'(0) = -2.5 c_1 + 0.5 c_2 \qquad y'(0) = -2.0$

$\Rightarrow 0.5 C_2 - 2.5 c_1 = -2$

~~$c_1 = 1.5$~~ ~~$-0.5C_2 = 2.5C_1$~~ $\qquad c_1 = 1.5$

~~$0.5C_2 = 2.5 \times 1.5$~~

~~$C_2 = \frac{3.75}{0.5}$~~

$0.5 C_2 - 2.5 \times 1.5 = -2$

$0.5 C_2 - 3.75 = -2$

$0.5 C_2 = 1.75$

$C_2 = 3.5$

$\therefore$ The G.S ie ~~$y = 1.5 e^{-2.5x}[1$~~

$y = e^{-2.5x}[1.5 \cos 0.5x + 3.5 \sin 0.5x]$

①

Differentiating ① w. r. to x

$y = 1.5\, e^{-2.5x} \cos 0.5x + 3.5\, e^{-2.5x} \sin 0.5x$

$\Rightarrow y' = 1.5\left[\left(e^{-2.5x} \times -\sin 0.5x \times 0.5 + \cos 0.5x \times -2.5 \times e^{-2.5x}\right] + 3.5\left[e^{-2.5x} \times \cos 0.5x \times 0.5 + \sin 0.5x \times -2.5 \times e^{-2.5x}\right]$

$\Rightarrow y' = 1.5\left(-0.5\, e^{-2.5x} \sin 0.5x - 2.5\, e^{-2.5x} \cos 0.5x\right) +$
$3.5\left(0.5\, e^{-2.5x} \cos 0.5x - 2.5\, e^{-2.5x} \sin 0.5x\right)$

$\Rightarrow y' = -0.75\, e^{-2.5x} \sin 0.5x - 3.75\, e^{-2.5x} \cos 0.5x +$
$1.75\, e^{-2.5x} \cos 0.5x - 8.75\, e^{-2.5x} \sin 0.5x.$

$y' = \left(-0.75 \sin 0.5x - 3.75 \cos 0.5x + 1.75 \cos 0.5x - 8.75\, \underline{\sin 0.5\, x}\right) e^{-2.5x}$

$y' = \left[\left(-0.75 - 8.75\right) \sin 0.5x + \left(-3.75 + 1.75\right) \cos 0.5x\right] e^{-2.5x}$

$y' = e^{-2.5x}\left(-9.5 \sin 0.5x - 2 \cos 0.5x\right)$

$y' = -e^{-2.5x}\left(9.5 \sin 0.5x + 2 \cos 0.5x\right)$

$\Rightarrow y' = -9.5\, e^{-2.5x} \sin 0.5x - 2\, e^{-2.5x} \cos 0.5x.$

$y'' = -9.5\left[e^{-2.5x} \times \cos 0.5x \times 0.5 + \sin 0.5x \times -2.5 \times e^{-2.5x}\right]$
$- 2\left[e^{-2.5x} \times -\sin 0.5x \times 0.5 + \cos 0.5x \times -2.5 \times e^{-2.5x}\right]$

$\rightarrow y'' = -9.5 \times 0.5\, e^{-2.5x} \cos 0.5x + 9.5 \times 2.5\, e^{-2.5x} \sin 0.5x$
$+ 2 \times 0.5\, e^{-2.5x} \sin 0.5x + 2 \times 2.5 \times e^{-2.5x} \times \cos 0.5x$

$$y'' = -4.75e^{-2.5x}\cos 0.5x - 23.75e^{-2.5x}\sin 0.5x +$$
$$1\cdot e^{-2.5x}\sin 0.5x + 5e^{-2.5x}\cos 0.5x$$

$$y'' = (-4.75\cos 0.5x - 23.75\sin 0.5x +$$
$$1\sin 0.5x + 5\cos 0.5x)$$

$$y'' = ((-4.75+5)\cos 0.5x + (-23.75+1)\sin 0.5x / e$$

$$\underline{\underline{y'' = 0.25\times e^{-2.5x}\times\cos 0.5x - 22.75\times\sin 0.5x\times e}}$$

4.ω <u>Reduction of</u> order

9) $x^2y'' + xy' - 4y = 0$

By guess

$y_1 = x^2$, $y_1' = 2x$, $y_1'' = 2$

$\Rightarrow$

$\therefore 2x^2 + 2x^2 - 4x^2 = \underline{\underline{0}}$

$y_1 = x^2$ is a solution of the given D.E

i.e, First basis.

Then we have to find the 2nd base

$y_2 = vy_1$, $v = \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx$

Rearranging the given equation dividing through x^2

$$\frac{x^2}{x^2}y'' + \frac{xy'}{x^2} - \frac{4}{x^2}y = 0$$

$$\Rightarrow y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0 \quad \text{①}$$

Comparing ① with standardequation

$$y'' + p(x)y' - q(x)y = 0$$

$\therefore p = \frac{1}{}$

$$\therefore \int p\,dx = \int \frac{1}{x}\,dx$$

$$= \underline{\underline{\log(x)}}$$

$$e^{-\int p\,dx} = e^{-\log x} = \underline{\underline{\frac{1}{x}}}.$$

$$\therefore V = \int \frac{1/x}{x^{4}}$$

$$= \int \frac{1}{x^{5}}$$

$$= \int x^{-5} = \frac{x^{-5+1}}{-5+1} = \underline{\underline{\frac{x^{-4}}{-4}}}$$

$$y_2 = V y_1$$

$$y_2 = \left(-\frac{1}{4}\, x^{-4}\right) \cdot x^{2}$$

$$= -\frac{1}{4} \times x^{-4+2}.$$

$$= \underline{\underline{-\frac{1}{4} \times x^{-2}}}$$

20/1/16

HW

tutorial 2 (2)

→ (ans $\lambda = -1, \pm 2i$)

30/1/16 Existence and uniqueness theoram for
Saturday Initial Value Problem (second Order)

Consider the Initial Value problem (I.V.P)

$$y'' + p(x)y' + q(x)y = 0 \quad \& \quad y(x_0) = y_0, \ y'(x_0) = y'_0$$

If ① $p(x)$ & $q(x)$ are continuous function on an interval 'I' and

② $x_0 \in I$, then the I VP; has a unique solution $y(x)$ on 'I'

- existence → atleast 1 or solution [illegible]
- uniqueness → atmost 1 [illegible] (only 1 solution)

Wrongskain

The wrongskain of y_1 and y_2 is defined by

$$w = w(y_1; y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Eg: Let $y_1 = e^x$, $y_2 = \cos x$, the wrongskai

$$W = w(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix}$$

$$= -e^x \sin x - e^x \cos x$$

$$= -e^x \underline{(\sin x + \cos x)}$$

- Theoram

The properties of wrongskain

① If y_1 and y_2 are any two solutions of the homogeneous LDE, then their Wrongskaing

$\omega = \omega(y_1, y_2)$ is identically zero, or never zero on 'I'.

② If y_1 & y_2 are the two solutions of the homogeneous LDE, on 'I', then they are linearly dependent iff their wrongskain $w(y_1, y_2)$ is identially zero on I $\left\{ \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0 \right\}$ and linearly independent $\omega(y_1, y_2)$ are not equal to zero

① Test the dependency and independency for the two solution, $y_1 = \cos \omega x$, & $y_2 = \sin \omega x$ of the D.E: $y'' + \omega^2 y = 0$.

ans: To check the dependency and independency, we have to calculate the Wrongskain

$$W = \omega(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix}$$

$$= \cos \omega x \times \omega \cos \omega x - \sin \omega x \times -\omega \sin \omega x.$$

$$= \omega \cos^2 \omega x + \omega \sin^2 \omega x$$

$$= \omega(\cos^2 \omega x + \sin x) = \underline{\underline{\omega}} \neq 0.$$

the given D.E is independent.

② Show that $y = (c_1 + c_2 x)e^x$ is the General Solution (G.S) of the D.E, $y'' - 2y' + y = 0$ on any interval

ans: Given, $y = (c_1 + c_2 x) e^x$

$$y = c_1 e^x + c_2 x e^x$$

$$y = c_1 y_1 + c_2 y_2$$

$y_1 = e^x$, $y_2 = xe^x$

Let e^x and xe^x be two solutions of
which is given.

$y_1 = e^x$

$y_1' = e^x$

$y_1'' = e^x$

$\therefore y_1'' - 2y_1' + y_1 = e^x - 2e^x + e^x = \underline{\underline{0}}$

$\therefore y_1 = e^x$ is a solution of the given D.E

$y_2 = xe^x$

$y_2' = x \times e^x + e^x \times 1 = e^x($

$y_2' = e^x + xe^x$

$y_2'' = e^x + (x \times e^x + e^x \times$

$y_2'' = e^x + xe^x + e^x$

$y_2'' = 2e^x + xe^x$

$y_2'' = \underline{\underline{(x+2)e^x}}$

$y_2'' - 2y_2' + y_2 = e^x(x+2) - 2e^x(x+1) + xe$

$= e^x x + 2e^x - 2e^x x - 2e^x + xe^x$

$= \underline{\underline{0}}$

$\therefore y_2 = xe^x$ is a solution of the given D

$\therefore y_1$ & y_2 are the solutions of the given D

Now, for ~~the~~ y_1 and y_2, their Wrongskain

$$w(y_1, y_2) = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix}$$

$= e^x \times (xe^x + e^x) - xe^x \times e^x$

$= xe^{2x} + e^{2x} - xe^{2x}$

$= \underline{\underline{e^{2x}}} \neq 0 \quad \therefore y_1$ & y_2 are linearly ~~The given D.E is~~ independent.

$\therefore y = c_1 y_1 + c_2 y_2 \Rightarrow y = c_1 e^x + x c_2 e^x$

Theorem

③ If p(x) and q(x) are continuous function on an open interval, I, then $y''+p(x)y'+q(x)y=0$ has a G.S, on 'I'

Qus ③ show that the following solutions are linearly independent.

(i) $e^{-0.5x}$ $e^{-2.5x}$

$y_1 = e^{-0.5x}$, $y_2 = e^{-2.5-x}$

$$w = w(y_1, y_2) = \begin{vmatrix} e^{-0.5x} & e^{-2.5x} \\ -0.5\,e^{-0.5x} & -2.5\,e^{-2.5x} \end{vmatrix}$$

$$= e^{-0.5x} - 2.5 e^{-2.5} - e^{-2.5} x - 0.5 e^{-0.5}$$

$$= -2e^{-3x} \neq 0$$ linearly independent.

HW
Tutorial 2

∴ The given solution, $y_1 = \cos 5x$ & $y_2 = \sin 5x$ are linearly independent.

By Fundamental theorem, we have

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 \cos 5x + c_2 \sin 5x \quad \text{①}$$

Differentiating ① with resp. to 'x'.

~~$y(0) = 3$.~~

~~do~~ $y' = -5c_1 \sin 5x + 5c_2 \cos 5x \quad \text{②}$

Differentiating ② w.r to x.

$$y'' = -25c_1 \cos 5x - 25c_2 \sin 5x$$

$$y'' = -25(c_1 \cos 5x + c_2 \sin 5x)$$

~~$y'' + 25 = \frac{25}{2} c_1 \cos 5x + c_2 \sin 5x$~~

$y'' = -25y_1$ from ① $\Rightarrow$

~~$y'' + 25 = y$~~

$$\therefore \quad y'' + 25y_1 = 0$$

$y'' + 25y$ is the required D.E

Thus the IVP is $y'' + 25y = 0$

$y(0) = 3 \qquad y'(0) = -5$

we have the G.S, $y(x) = c_1 \cos 5x + c_2 \sin 5x$

~~also~~, Apply to ①

a $y = c_1 \cos 0 + c_2 \sin 0 \Rightarrow y = c_1$

$y(0) = 3 \Rightarrow \underline{\underline{c_1 = 3}}$

from ② $\Rightarrow$

$y'(x) = -5c_1 \sin 5x + 5c_2 \cos 5x$

$y'(0) = -5 \Rightarrow$

$y'(0) = -5c_1 \sin 0 + 5c_2 \cos 0$

$y'(0) = 5c_2 \Rightarrow 5c_2 = -5 \Rightarrow \underline{\underline{c_2 = -1}}$

Then the particular solution, y_p ie

$y = 3 \cos \underline{\underline{5x -}} \sin 5x$

(b) $\cos h\, 1.8x, \quad \sin h\, 1.8x \qquad I.V \rightarrow y(0) = 14.2,$

$y'(0) = 16.38$

ans: $y_1 = \cos h\, 1.8x, \qquad y_2 = \sin h\, 1.8x.$

$\cosh^2 x - \sinh^2 x = 1$

$\frac{d}{dx} \sinh x = \cosh x$

$\frac{d}{dx} \cosh x = \sinh x$

$$w(y_1, y_2) = \begin{vmatrix} \cos h\, 1.8x & \sin h\, 1.8x \\ 1.8 \sin h\, 1.8x & 1.8 \cos h\, 1.8x \end{vmatrix}$$

$= \cancel{\cos h^2\, 1.8x + \sin h^2\, 1.8}$

$= \cos h\, 1.8x \times 1.8 \cos h\, 1.8x - \sin h\, 1.8x \times 1.8 \times \sin h\, 1.8x$

$= 1.8 \cos h^2\, 1.8x - 1.8 \sin h^2\, 1.8$

$= 1.8\, (\cos h^2 x - \sin h^2 x) = \underline{\underline{1.8}} \neq 0$

the given solution

$y_1 = \cos h\, 1.8x \qquad y_2 = \sin h\, 1.8x$

are linearly independent

By Fundamental theorem $\quad y = C_1 y_1 + C_2 y_2$

$y = C_1 \cos h(1.8x) + C_2 \sin h(1.8x)$ ——①

Differentiating w. r. to x

$y' = 1.8\, C_1 \sin h(1.8x) + 1.8\, C_2 \cos h(1.8x)$ —②

Differentiating again w. r. to x

$y'' = 3.24\, C_1 \cos h(1.8x) + 3.24\, C_2 \sin h(1.8x)$

$y'' = 3.24\, (C_1 \cos h(1.8x) + C_2 \sin h(1.8x)$

from ① $\Rightarrow \quad y'' = 3.24y$

$y'' - 3.24\, y = 0$

$y'' - 3.24\, y$ is the required D.E

IVP is $\quad y(0) = 14.2$

① $\Rightarrow \quad y = C_1 \cos h(1.8x) + C_2 \sin h(1.8x)$

$C_1 = 14.2$

$y'(0) = 16 \cdot 38$

from ② ⇒ $y'(x) = 1 \cdot 8\, C_1 \sinh(1 \cdot 8x) + 1 \cdot 8\, C_2 \cosh($

$y'(0) = 1 \cdot 8\, C_2 \Rightarrow 1 \cdot 8\, C_2 = 16 \cdot 38$

$$C_2 = \frac{16 \cdot 38}{1 \cdot 8} = \underline{\underline{9 \cdot 1}}$$

$\therefore C_1 = 14 \cdot 2, \quad C_2 = 9 \cdot 1$

∴ The particular solution ie y_p i.e

$$y = 14 \cdot 2 \underline{\underline{\cosh(1 \cdot 8x)}} + 9 \cdot 1 \sinh(1 \cdot 8x)$$

Tutorial2

HW ⑦ (i) $e^{-2 \cdot 5x} \cos 0 \cdot 5x$, $e^{-2 \cdot 5x} \sin 0 \cdot 5x$

$y(0) = 1 \cdot 5, \quad y'(0) = -2$

(ans: $w = 5e^{-3x}$, $y'' + 5y' + 6 \cdot 5y = 0$)

(ii) $1, e^{3x}, \quad y(0) = 2, \quad y'(0) = -1$

(ans: $w = 3e^{3x}$, $y'' - 3y' = 0$)

1/2/16 linear independency & dependency of higher orders

consider the 'n' solution function $y_1, y_2, \ldots y_n$ defin on an open interval I. These functions are called linearly independent on I. If the ~~eqn~~ equation $k_1y_1 + k_2y_2 + k_3y_3 + \ldots + k_ny_n = 0$ where all $k_1, k_2, k_3 \ldots k_n$ all are zero

These functions are said to be linearly dependen if the equations $k_1y_1 + k_2y_2 + k_3y_2 \ldots k_ny_n = 0$ where not all $k_1, k_2, k_3, \ldots k_n$ =zero is equal to zero

1) show that the solution function $y_1 = x$ $y_2 = x^2$, $y_3 = x^3$ are Linearly independent on any interval $-1 \leq x \leq 2$

According to the conditions of independency $k_1 x + k_2 x^2 + k_3 x^3 = 0$

$\therefore x = -1$, $-k_1 + k_2 - k_3 = 0$ — ①

$x = 1$ $k_1 + k_2 + k_3 = 0$ — ②

$x = 2$ $2k_1 + 4k_2 + 8k_3 = 0$ — ③

$\therefore$ The given functions are linearly independent.

2) show that the function $y_1 = e^{-3x}$, $y_2 = \cos 2x$ $y_3 = \sin 2x$ are Linearly independent.

$y_1 = e^{-3x}$, $y_2 = \cos 2x$, $y_3 = \sin 2x$

$$w = w(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \quad e^{-3x}$$

$$= \begin{vmatrix} e^{-3x} & \cos 2x & \sin 2x \\ -3e^{-3x} & -2\sin 2x & 2\cos 2x \\ 9e^{-3x} & -4\cos 2x & -4\sin 2x \end{vmatrix}$$

$$= e^{-3x} \begin{vmatrix} -2\sin 2x & 2\cos 2x \\ -4\cos 2x & -4\sin 2x \end{vmatrix} - \cos 2x \begin{vmatrix} -3e^{-3x} & 2\cos 2x \\ 9e^{-3x} & -4\sin 2x \end{vmatrix} + \sin 3x \begin{vmatrix} -3e^{-3x} & -2\sin 2 \\ 9e^{-3x} & \cancel{-4\cos 2} \; -4\cos 2x \end{vmatrix}$$

$$= e^{-3x}(8\sin^2 2x + 8\cos^2 2x) - \cos 2x\,(12e^{-3x}\sin 2x + 18e^{-3x}\cos 2x$$

$$+ \sin 2x\,(12e^{-3x}\cos 2x - 18e^{-3x}\sin 2x)$$

$$= 8e^{-3x}(\sin^2 x + \cos^2 x) - 12e^{-3x}\sin 2x\cos 2x -$$

$$18e^{-3x}\cos^2 2x + \sin 2x\,12e^{-3x}\sin 2x\,(\cos 2x + 18e^{-3x}\sin$$

$$= 8e^{-3x} + 18e^{-3x}(\sin^2 x + \cos^2 x)$$

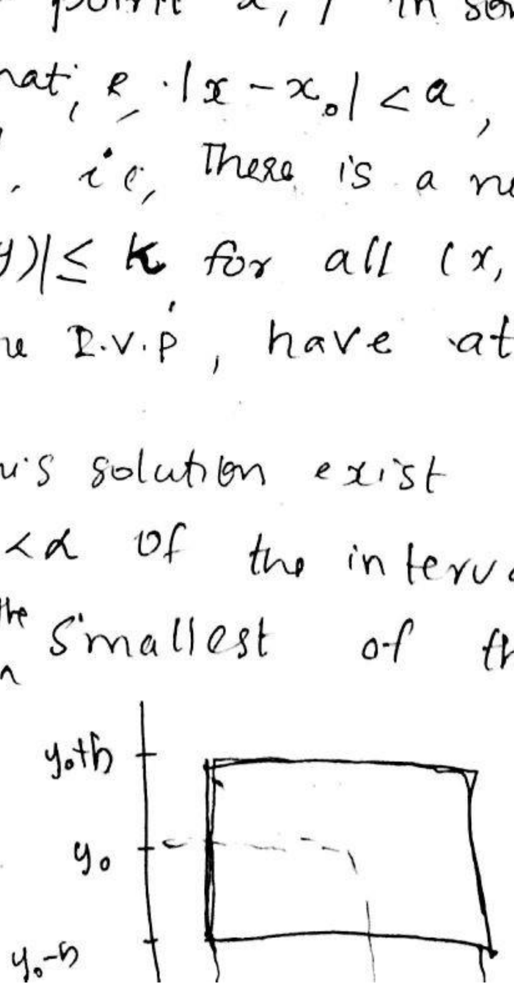
$$= \underline{26e^{-3x}} \neq 0$$

Existence Theorem

Let the right side $f(x,y)$ of the first order ODE IVP, $y' = f(x,y)$ when $y(x_0) = y_0$ be continuous at all point x, Y in some rectangle 'R' such that, R $|x - x_0| < a$, $|y - y_0| < b$ are bounded in 'R', i.e, There is a no. 'K' such that $|F(x,y)| \leq K$ for all (x,y) in 'R'.

Then the I.V.P, have at least one solution $y(x)$.

Then this solution exist at least one $\forall$ all $|x - x_0| < \alpha$ of the interval $|x - x_0| < a$ here 'α' is the smallest of the two nos, a & $\frac{b}{K}$.



Uniquenees theorem

Let 'f' and partial derivative $f_y \cdot \frac{\partial f}{\partial y}$ is continuous ∀ all (x,y) in the rectangle and bounded

(a) $|f(x,y)| \le k$,

(b) $|f_y(x,y)| \le M$ ∀ (x,y) in R.

Then the I.V.P, $y' = f(x,y)$ where $y(x_0) = y_0$ has at most one solution $y(x)$. By Existence theorm the problem, precisely one solution, this solution exist at least for all 'x' in that subintervel $(x - x_0) < \alpha$

Choice of Rectangle

consider the I.V.P $y' = 1 + y_2$, $y(0) = 0$

$R : |x| \le 5$, $|y| < 3$

From the given problem we have,

$a = 5$, $b = 3$ and $f(x,y) = 1 + y^2$

$|f(x,y)| = |1 + y^2| \le k$ $\quad 1+3^2 = 10$

$|f(x,y)| = |1 + y^2| \le = 10$

$|f_y(x,y)| = |2y| = M = 6$

$(\cdot 3 < a \& b)$

$$\alpha = \frac{b}{k} = \frac{3}{10} = \underline{\underline{\cdot 3}}$$

The n^{th} order non-homogeneous L.O.D.E of the following form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_n y = g(x) \quad \textcircled{1}$$

where $a_1, a_2 \ldots a_n$ & $g(x)$. ~~are~~ are fun of x

A Linear D.E has constant coefficient in the above equation all $a_1, a_2, \ldots, a_n$ are zero are of constant coefficient By the helpe of the differential operators

$$\frac{d}{dx} = D, \frac{d^2}{dx^2} \ldots D^2, \frac{d^n}{dx^n} = D^n$$

The above ① is written in the form

$$[D^n - a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n] y = g(x)$$

$$\boxed{F(D) y = 0}$$

where $f(D)$ is a polynomial in D.

2/2/2016 Tuesday

Particular Integral (P.I)

From the above equation, the particular integral P.I, $y = \frac{1}{f(D)} g(x)$

- The Complete solution (C.S) $y = C.F + PI$

Depending upon the nature of $g(x)$ we have to write the P.I. by using different cases.

Case 1

- when $g(x)$ of the form e^{ax}
Here the particular Integral (P.I),

$$y = \frac{1}{F(D)} g(x) = \frac{1}{F(D)} e^{ax}$$

Replaying $\frac{D}{a}$ $\quad D = a[\ ]$

provide $\quad = \frac{1}{f(a)} e^{ax}, \quad f(a) \neq 0$

- If $f(a) = 0$, then the case of failure,

$$P.I ;\ y = x \cdot \frac{1}{F'(a)} e^{ax}, \quad f'(a) \neq 0$$

- If $f'(a) = 0$ then again the case of failure

$$P.I,\ y = x^2 \frac{1}{f''(a)} \cdot e^{ax}, \quad f''(a) \neq 0$$

- If $f''(a) = 0$, then again the case of failure

$$P.I,\ y = x^3 \frac{1}{f'''(a)} \cdot e^{ax}, \quad f'''(a) \neq 0$$

and so on.

1) Solve $\frac{d^4y}{dx^4} - \frac{2d^3y}{dx^3} + 5\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$

ans:

S.F $\Rightarrow (D^4 - 2D^3 + 5D^2 - 8D + 4)y = e^x$

AE $\Rightarrow \lambda^4 - 2\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

$\lambda = 0, \quad f(0) = 4 \neq 0$

$\lambda = 1 \quad f(1) = 1 - 2 + 5 - 8 + 4 = 0$

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & 5 & -8 & 4 \\ & 0 & 1 & -1 & 4 & -4 \\ \hline & 1 & -1 & 4 & -4 & \boxed{0} \end{array}$$

$\lambda^3 - \lambda^2 + 4\lambda - 4 = 0$

$\lambda = 0, \; f(0) = -4 \neq 0$

$\lambda = 1 \; f(1) = 1 - 1 + 4 - 4 = \underline{\underline{0}}$

$$\begin{array}{r|rrrr} 1 & 1 & -1 & 4 & -4 \\ & 0 & 1 & 0 & 4 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4 \quad \Rightarrow \quad \lambda = \underline{\underline{\pm 2i}}$

The roots are $\lambda = 1, 1, \pm 2i$

$\alpha = 0, \beta =$

∴ The C·F i.e ~~$y = C_1 e^x + C_2 e^x$~~, e

$y = (C_1 + C_2 x) e^x + e^{0x} (C_3 \cos 2x + C_4 \sin 2x)$

∴ The G.S, i.e, $y = (C_1 + C_2 x) e^x + (C_3 \cos 2x + C_4 \sin 2x$

∴ The particular Integral i.e {P.I} i.e

$y = \frac{1}{F(D)} g(x)$

$F(D) = D^4 - 2D^3 + 5D^2 - 8D + 4 \quad , g(x) = e^x$

$y = \frac{1}{D^4 - 2D^3 + 5D^2 - 8D + 4} e^x$

$e^{ax} = e^x$

$(D = a = 1)$

$$\therefore y = \frac{1}{1^4 - 2\times1^3 + 5\times1^2 - 8\times1 + 4}\, e^{x}.$$

$$= \frac{1}{0}\, e^{x}$$, case of failure

- PCI ~~ie~~, y,

Differentiation the denominator of P.I (first)

$$\therefore P.I \text{ ie } y = x \times \frac{1}{4D^3 - 6D^2 + 10D - 8}\, e^{x} \qquad (D = a = 1)$$

Replacing D = a = 1

$$= x \times \frac{1}{4\times1^3 - 6\times1^2 + 10\times1 - 8}$$

$$= x \cdot \frac{1}{0} e^{x}$$, again the case of failure

Differentiationg again

$$P.I \text{ ie, } y = x^2 \frac{1}{12D^2 - 12D + 10}\, e^{x}$$

$$= x^2 \frac{1}{12\times1^2 - 12\times1 + 10}\, e^{x}$$

$$= x^2 \cdot \frac{1}{10} e^{x}$$

∴ The Complete Solution (C.S) ie,

C.S. ie, $y = C.F + P.I$

$$y = (C_1 + C_2 e^{x}) + (C_3 \cos 2x + C_4 \sin 2x)$$
$$+ x^2 \cdot \frac{1}{10} e^{x}$$

② Find the solution of the D·E

$(D^2+3D+2)y = 5$

ans: S.F ⇒ $(D^2+3D+2)\,y = 0$

A·E ⇒ $\lambda^2+3\lambda+2 = 0 \to \lambda^2+3\lambda+2=0$

$$\lambda = \frac{-3\pm\sqrt{3^2-4\times1\times2}}{2\times1} = \frac{-3\pm\sqrt{9-4\times2}}{2}$$

$$= \frac{-3\pm\sqrt{9-8}}{2} = \frac{-3\pm1}{2} = \frac{-3+1}{2} \text{ or } \frac{-3-}{2}$$

$$= \frac{-2}{2} \text{ or } \frac{-4}{2}$$

$$= \underline{-1} \text{ or } \underline{-2}$$

roots $\lambda = -1, -2$

$CF = c_1e^{-x}+c_2e^{-2x}.$

P.I ie $y = \frac{1}{F(D)}\,g(x) = \frac{1}{D^2+3D+2}\,5e^{0x}$

(D=a=

$$= \frac{1}{0^2+3\times0+2}\times5\times e^{0x}$$

$$= \underline{\frac{5}{2}e^{0x}}$$

C.S = CF + PI

C.S ie $y = c_1e^{-x}+c_2e^{-2x}+\frac{5}{2}$

again $P.I = \frac{1}{F(D)} g(x)$

$= \frac{1}{F(D^2)} \cos ax \; \boxed{\text{or}} \; \sin ax.$

(Replacing $D^2 = -a^2$)

$= \frac{1}{F(-a^2)} \cos ax \; \boxed{\text{or}} \; \sin ax, \; F(-a^2) \neq 0$

If $F(-a^2) = 0$, the case of failure

$P.I = x \cdot \frac{1}{F'(-a^2)} \cos ax \; \boxed{\text{or}} \; \sin ax,$

$F'(-a^2) \neq 0$

If then the case of failure

$P.I = x^2 \cdot \frac{1}{F''(-a^2)} \cos ax \; \boxed{\text{or}} \; \sin ax$

provided $f''(-a^2) \neq 0$ and so on.

③ Solve $(D^3+1)y = \cos 2x$

$(a^3+b^3) = (a+b)(a^2-ab+b^2)$

$(\lambda+1)(\lambda^2-\lambda+1)$

SF ⇒ $(D^3+1)y = 0$

HW CF

AE ⇒ $\lambda^3+1 = 0$

$\lambda^3 + 1 = 0$ ⇒

$\lambda^3 = -1 \Rightarrow \underline{\lambda = -1}$

$\lambda = 0 \quad f(0) = 0$

roots $\lambda = -1, \; f(-1) = -1+1 = 0$

$\lambda^2 - \lambda + 1 \Rightarrow \lambda = \frac{1 \pm \sqrt{1-4\times1\times1}}{2\times1}$

$\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \quad (\alpha = 1/2, \beta = \frac{\sqrt{3}}{2}$

roots are $\lambda = -1, \; \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

C.F i.e y_c i.e

$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$

$$P.I = \frac{1}{F(D^2)}\, g(x) \qquad F(D^2) = D^3+1$$
$$g(x) = \cos 2x)$$

$$= \frac{1}{D^3+1} \cos 2x \quad \text{(Replacing,} \quad D^2 = -a^2 = -2^2 = -4)$$

$$= \frac{1}{D(D^2)+1} \cos 2x$$

$$= \frac{1}{D\times -4+1} \cos 2x$$

$$= \frac{1}{-4D+1} \cos 2x$$

$$= \frac{1}{1-4D} \cos 2x$$

Multiplying both numerator & denominator by(1+

$$P.I = \frac{1\,(1+4D)}{(1-4D)(1+4D)} \cos 2x$$

$$= \frac{1+4D}{1^2-16D^2} \cos 2x \quad (D^2 = -a^2 = -2^2 = -4$$

$$= ~~1+4\times 4~~$$

$$\frac{(1+4D)}{1-16\times -4} \cos 2x$$

$$= \frac{1}{65} \times \cos 2x + 4D(\cos 2x)$$

$$= \frac{\cos 2x + 4(-2\sin 2x)}{65}$$

$$= \frac{\cos 2x - 8\sin 2x}{65}$$

$\therefore C.S = C.F + P.I$

~~C.S~~ =

$\therefore$ ~~y_p~~

$$y = C_1 e^{-x} + e^{\frac{1}{2}x}\left(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{\cos 2x - 8}{65}$$

[illegible] solve the D.E $\frac{d^4y}{dx^4} - m^4y = \sin mx$

ans: SF $\Rightarrow (D^4 - m^4)y = 0$

A.E $\Rightarrow \lambda^4 - m^4 = 0$.

C.F: $\lambda = 0 \rightarrow f(0) = -m^4 \neq 0$

$\lambda = m \Rightarrow f(m) = m^4 - m^4 = \underline{0}$.

$\lambda = 0$ is a root.

$$\begin{array}{c|ccccc} 0 & 1 & 0 & 0 & 0 & -1 \\ & 0 & 0 & 0 & 0 & 0 \\ \hline & 1 & 0 & 0 & 0 & \end{array}$$

$\lambda^4 - m^4 = 0 \Rightarrow (\lambda^2 - m^2)(\lambda^2 + m^2) = 0$

$\lambda^2 - m^2 = 0$ $\qquad \lambda^2 + m^2 = 0$

$\lambda^2 = m^2$ $\qquad \lambda^2 = -m^2$

$\lambda = \pm m$, $\qquad \lambda = \pm mi$ $(\alpha = 0, \beta = m)$

$\therefore CF = c_1 e^{mx} + c_2 e^{-mx} + e^{0x}(c_3 \cos mx + c_4 \sin mx)$

$P.I = \frac{1}{F(D^2)} g(x)$.

$= \frac{1}{F(D^2)} \sin mx$.

$= \frac{1}{D^4 - m^4} \sin mx$ $\quad \left(D^2 = -(a^2) = -m^2 \right.$
$\left. D^4 = m^4 \right)$

$= \frac{1}{m^4 - m^4} \sin mx$

$= \frac{1}{0} \sin mx$, the case of failure.

$$P \cdot I = x \cdot \frac{1}{f'(D^2)} \sin mx$$

$$= x \cdot \frac{1}{4D^3} \cdot \sin mx \qquad (D^2 = -a^2 = -$$

$$= x \cdot \frac{1}{4D(-m^2)} \sin mx$$

$$= \frac{x \cdot}{-4m^2} \cdot \frac{1}{D} (\sin mx)$$

Inform $\int \sin mx = \frac{-\cos mx}{m}$

$$= \frac{x}{-4m^2} \int \sin mx \, dx$$

$$= \frac{-x}{-4m^2} \times \frac{-\cos mx}{m}$$

$$= \underline{\underline{\frac{x \cos mx}{4m^3}}}$$

$$\therefore C \cdot S = CF + P \cdot I$$

$$y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + \frac{x \cos mx}{4m3}$$

5) Solve ~~[illegible]~~

binomial theorem for a negative index is applicable)

The remaining factor will be of the form $[1+\phi(D)]$. or $[1-\phi(D)]$

Take this factor to the numerator and

takes of the form $[1+\phi(D)]^{-1}$ or $[1-\phi(D)]$

Then expand it in assending powers of 'D' as far as the term containing D^m,

Since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$ and so on.

① Solve $(D^2+2)y = x^2$

ans: SF ⇒ $(D^2+2)y = 0$

A·E ⇒ $\lambda^2 + 2 = 0$

$\lambda^2 = -2 \Rightarrow \lambda = \pm i\sqrt{2}$. $(\alpha = 0, \beta = \pm\sqrt{2})$

C·F i.e y_c i.e $y = e^{0x}(c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x)$

$y = c_1 \cos\sqrt{2}\,x + c_2 \sin\sqrt{2}\,x$.

$P.I = \frac{1}{F(D)} x^m$ $\quad g(x) = x^m$

$= \frac{1}{D^2+2} x^2$ (taking the lowest degree term common)

$= \frac{1}{2\left(\frac{D^2}{2}+1\right)} x^2$

$= \frac{1}{2\left(1+\frac{D^2}{2}\right)} x^2$

$= \frac{1}{2}\left(1+\frac{D^2}{2}\right)^{-1} \cdot x^2$

$$P.I = \frac{1}{2}\left[1 - \frac{D^2}{2} + \frac{D^4}{4} \dots\right] x^2$$

$\boxed{\begin{array}{l} * \ (1+x)^{-1} = 1 - x + x^2 \\ \text{Here, } x = \frac{D^2}{2} \end{array}}$

$\boxed{x^2 = D^2}$

$$P.I = \frac{1}{2}\left[x^2 - \frac{1}{2} D^2(x^2)\right]$$

$$= \frac{1}{2}\left(x^2 - \frac{1}{2} \times 2\right)$$

$$= \frac{1}{2}\left(x^2 - 1\right)$$

$\boxed{\begin{array}{l} D(x^2) = 2x \\ D^2(x^2) = 2 \end{array}}$

$* \ \boxed{(1-x)^{-1} = 1 + x + x^2 + }$

$C.S = C.F + P.I$

i.e

$$y = C_1 \cos\sqrt{2}\,x + C_2 \sin\sqrt{2}\,x + \frac{x^2 - 1}{2}$$

② Solve, $(D^3 - 8)y = x^4 + 2x + I + \cosh 2x$

$C.F \Rightarrow (D^3 = 8)y = 0$

$A.E \Rightarrow \lambda^3 - 8 = 0 \qquad \lambda^3 = 2^3$

$2 \times 2 \times 2$

$\lambda^3 - 2^3 = 0$ ~~and~~

$\lambda^3 - 2^3 = (\lambda - 2)(\lambda^2 + 2\lambda + 2^2)$

$\boxed{\begin{array}{l} a^3 - b^3 = \\ (a-b)(a^2 + ab + b \end{array}}$

$\lambda = 2 . \qquad \lambda^2 + 2\lambda + 4 = 0$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \times 1 \times 4}}{2 \times 1}$$

$1 - 32$

$$= \cancel{\frac{-2 \pm \sqrt{-32}}{2}} = \cancel{\frac{-2 + \sqrt{32}\,i}{2}} \text{ or } \cancel{\frac{-2 -}{2}}$$

$$\lambda = \frac{-2 \pm \sqrt{4 - 16}}{2} = \frac{-2 \pm \sqrt{-12}}{2}$$

$2 \,|\, 12$
$2 \,|\, 6$
3

$$= \frac{-2 + 2\sqrt{3}\,i}{2} \text{ or } \frac{-2 - 2\sqrt{3}\,i}{2}$$

$$\lambda = -1 + \sqrt{3}\,i \ \& \ -1 - \sqrt{3}\,i$$

$\text{v.F} \Rightarrow x \overset{\text{roots}}{=} 2 \quad -1 \pm \sqrt{3}i \quad (\alpha = -1,\ \beta = \sqrt{3})$

$\text{C.F} = c_1 e^{2x} + \underline{\underline{e^{-1x}}} \left(c_2 \cos\sqrt{3}x + c_3 \sin\sqrt{3}x \right)$

$\text{P.I} = (D^3 - 8)y = \underbrace{x^4 + 2x + I}_{P.I_1} + \underbrace{\cosh 2x}_{P.I_2},$ $\qquad I = 1$ $\quad x^0 \text{ or } e^{0x}$

$P.I_1 = \frac{1}{F(D)} g(x) \qquad (I = 1)$

$= \frac{1}{D^3 - 8} (x^4 + 2x + 1)$

$= \frac{1}{-8\left(-\frac{D^3}{8} + 1\right)} (x^4 + 2x + 1)$ $\qquad -8 - D^3$ $\quad -8\left(1 + \frac{D^3}{-8}\right)$ $\quad -8\left(1 - \frac{D^3}{8}\right)$

$= \frac{1}{-8\left(1 - \frac{D^3}{8}\right)} (x^4 + 2x + 1)$

$= -\frac{1}{8} \times \left(1 - \frac{D^3}{8}\right)^{-1} \cdot (x^4 + 2x + 1)$ hi.

$= \frac{-1}{8}\left[1 + \frac{D^3}{8} + \frac{D^6}{64}\right](x^4 + 2x + 1)$ $\qquad (1-x)^{-1} = 1 + x + x^2 + \cdots$ $\quad x^4 = D^4$

$= \frac{-1}{8}\left[x^4 + 2x + 1 + \frac{1}{8} D^3 (x^4 + 2x + 1)\right]$

$= \frac{-1}{8}\left[x^4 + 2x + 1 + \frac{1}{8} \times 24x\right]$

$= -\frac{1}{8}\left(x^4 + 2x + 1 + 3x\right.$

$= \frac{-1}{8}\left[(x^4 + 2x + 1) + 3x\right]$

$D(x^4 + 2x + 1) = 4x^3 + 2$

$D^2(x^4 + 2x + 1) = 12x^2$

$D^3(x^4 + 2x + 1) = 24x$

$$P.I_2 = \frac{1}{D^3-8}\cosh 2x$$

$$\boxed{\begin{array}{l}\cosh 2x = \dfrac{e^{2x}+e^{-2x}}{2} \\ \sinh 2x = \dfrac{e^{2x}-e^{-2x}}{2}\end{array}}$$

$$P.I_2 = \frac{1}{D^3-8}\left(\frac{e^{2x}+e^{-2x}}{2}\right)$$

$$= \frac{1}{2}\left(\frac{1}{D^3-8}\left(e^{2x}+e^{-2x}\right)\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^3-8}e^{2x} + \frac{1}{D^3-8}e^{-2x}\right]$$

Replacing

$$\boxed{\begin{array}{l} e^{2x} \\ (D=a=2) \\ e^{-2x} \\ (D=a=-2)\end{array}}$$

$$= \frac{1}{2}\left[\frac{1}{2^3-8}e^{2x} + \frac{1}{\cancel{2^3}-8-8}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{1}{0}e^{2x} + \frac{1}{-16}e^{-2x}\right]$$ case of failure

Differentiating the denominator, multiply x.

$$= \frac{1}{2}\left[x \cdot \frac{1}{3D^2}e^{2x} - \frac{1}{16}e^{-2x}\right]$$

$$(D=a=2)$$

$$= \frac{1}{2}\left[x \cdot \frac{1}{3\times 2^2}e^{2x} - \frac{1}{16}e^{-2x}\right]$$

$$= \frac{1}{2}\left[\frac{x}{12}e^{2x} - \frac{1}{16}e^{-2x}\right]$$

$$C.S = C.F + (P.I_1 + P.I_2)$$

$$C.S = C_1e^{2x} + e^{-x}(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x +$$

$$-\frac{1}{8}(x^4+5x+1) + \frac{1}{2}\left[\frac{x}{12}e^{2x} - \frac{1}{16}e^{-2x}\right)$$

re Find the P.I of $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4\cos^2 x$

$$P.I = \frac{1}{F(D)} g(x)$$

$$= \frac{1}{D^2+3D+2} \times 4\cos^2 x.$$

$$\boxed{\begin{array}{l}\cos 2x = \\ 2\cos^2 x - 1 \\ \cos^2 x = \frac{\cos 2x + 1}{2}\end{array}}$$

$$= 4 \cdot \frac{1}{D^2+3D+2} \cdot \left(\frac{\cos 2x + 1}{2}\right)$$

$\frac{4}{2} = 2$

$$= 2 \frac{1}{D^2+3D+2} (\cos 2x + 1)$$

$1 = e^{0x}$ or x^{0}

$$= 2 \frac{1}{D^2+3D+2} (\underbrace{\cos 2x}_{P.I_1} + \underbrace{e^{0x}}_{P.I_2})$$

~~Sin2x~~
~~cos~~ 2x =
~~1+2cos²x~~

$$P.I = 2 [P.I_1 + P.I_2] \quad \text{①}$$

$$\boxed{\begin{array}{l}\cos 2x = \\ 1 - 2\sin^2 x\end{array}}$$

$$P.I_1 = \frac{1}{D^2+3D+2} \cdot \cos 2x.)$$

Replacing, ($D^2 = -a^2 = -2^2 = -4$)

$$P.I_1 = \frac{1}{-4+3D+2} \cos 2x$$

$$P.I_1 = \frac{1}{3D-2} \cos 2x.$$

Multiplying both numerator & denominator by $(3D+2)$

$$P.I_1 = \frac{(3D+2)}{(3D-2)(3D+2)} \cos 2x.$$

$3^2D^2 - 2^2$
$9D^2 - 4$

$$= \frac{(3D+2)(\cos 2x)}{9D^2-4}$$

Replacing $D^2 = -a^2 = -4$

$$= (3D+2)\ (\cos 2x)$$

$3 \cdot 6 +$

$$= \frac{(3D+2)\,(\cos 2x)}{-40}$$

$D\cdot\cos 2x = 2\sin 2x$

$$= \frac{3D\cdot\cos 2x + 2\cos 2x}{-40}$$

$$= \frac{\cancel{e}3 \times 2\times -\sin 2x + 2\cos 2x}{-40}$$

$$\frac{-6\sin 2x + 2\cos 2x}{-40}$$

$$= \frac{\cancel{-}3\sin 2x \cancel{+} -\cos 2x}{20} \quad 2\,\cancel{\cos}$$

$$= \frac{1}{20}\left(3\sin 2x - \cos 2x\right)$$

$$P\cdot I_2 = \frac{1}{D^2+3D+2}\, e^{0x}. \qquad (D=a=0)$$

$$= \frac{1}{0+3\times 0+2} = \frac{1}{2}.$$

$$P\cdot I = 2\left[\frac{1}{20}(3\sin 2x - \cos 2x) + \frac{1}{2}\right)$$

$$= \frac{2}{20}\,\cancel{\$}(3\sin 2x - \cos 2x)\cdot + \frac{2}{2}$$

$$= \frac{1}{10}(3\sin 2x - \cos 2x) + 1$$

Aw

$2\sin \cancel{x} \cos y =$

4/1/2016 Solve $(D^3 - D^2 - 6D)y = x^2+1$

ans: $S.F \Rightarrow (D^3 - D^2 - 6D)y = 0$

$A.E \Rightarrow \lambda^3 - \lambda^2 - 6\lambda = 0$

$\lambda(\lambda^2 - \lambda - 6) = 0$

$\lambda = 0,$ $\lambda^2 - \lambda - 6 = 0$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times -6}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{1+24}}{2} = \frac{1 \pm \sqrt{25}}{2} = \frac{1+5}{2} \text{ \& } \frac{1-5}{2}$$

$$= \frac{6}{2} \text{ \& } \frac{-4}{2}$$

$$= \underline{\underline{3}} \text{ \& } \underline{\underline{-2}}$$

$\lambda = 0, \lambda = 3, \lambda = -2$

$$C.F = c_1 e^{0x} + c_2 e^{3x} + c_3 e^{-2x}$$

$$P.I = \frac{1}{F(D)} g(x).$$

$$= \frac{1}{D^3 - D^2 - 6D} (x^2+1)$$

$\frac{D^3}{D^2} \quad -6D \cdot \frac{D^3}{D}.$

$$= \frac{1}{-6D\left[\frac{D^3}{-6D} - \frac{D^2}{-6D} + 1\right)} (x^2+1)$$

$$= \frac{1}{-6D\left(1 + \left(\frac{D^3}{6D} - \frac{D^2}{6}\right)\right)} (x^2+1)$$

$$= -\frac{1}{6} \cdot \frac{1}{D} \left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1} (x^2+1)$$

~~x+1~~

$(1+x)^{-1} =$

$1 - x + x^2 - x^3$

$$= -\frac{1}{6} \cdot \frac{1}{D} \left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 - \cdots\right](x^2+1).$$

$x = \frac{D}{6} - \frac{D^2}{6}$ maximum power=2 (x^2+1)

$1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)$

$* \quad (a-b)^2 = D^2 - 2D^3 + D^4$

cancelling this term

$$= \frac{-1}{6} \cdot \frac{1}{D} \left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36}\right] \cdot (x^2+1)$$

$D^2(x^2+1) = 2$
$D^2(x^2+1) = ..$

9)

$$= \frac{-1}{6} \cdot \frac{1}{D}\left[(x^2+1) - \frac{1}{6} D(x^2+1) + \frac{1}{6} D^2(x^2+1) + \frac{1}{36} D^2(x^2+1)\right]$$

$$= \frac{-1}{6} \cdot \frac{1}{D}\left[(x^2+1) - \frac{1}{6} \times 2x + \frac{1}{6} \times 2 + \frac{1}{36} \times 2.\right]$$

$$= \frac{-1}{6} \cdot \frac{1}{D}\left[(x^2+1) - \frac{x}{3} + \frac{1}{3} + \frac{1}{18}\right]$$

$$= -\frac{1}{6} \frac{1}{D} \left(x^2 - \frac{1}{3} x + \frac{1}{3} + \frac{1}{18} + 1\right)$$

$\frac{18}{\frac{3}{21}}$

$\frac{1}{3} + \frac{1}{18} = \frac{18+3}{54}$

$= \frac{21}{54}$

$= \frac{21+54}{54}$

$\frac{22}{54}$

$$= -\frac{1}{6} \cdot \frac{1}{D} \left(x^2 - \frac{1}{3} x + \frac{25}{18}\right)$$

$$= -\frac{1}{6} \int x^2 - \frac{1}{3} x + \frac{25}{18} \, dx$$

$$= \frac{-1}{6} \left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25x}{18}\right]$$

$18 \times 6 = 108$

$$= \frac{-x^3}{18} + \frac{x^2}{36} - \frac{25x}{108}$$

∴ The C.S ie $y = CF + P \cdot I$

$$\therefore y = C_1 + C_2 e^{3x} + C_3 e^{-2x} - \frac{x^3}{18} + \frac{x^2}{36} - \frac{25x}{108}$$

Hw

7) $(D^3 + 2D^2 + D)y = e^{2x}$ 2

$2e^{2x}\cosh x = 2e$

9) Solve $(D^2-4D+4)y = \sin 2x$, given that $y = \frac{1}{8}$, $Dy = 4$. when $x = 0$. Find the value of 'y' when ~~$x = 0$?~~ $x = \frac{\pi}{4}$?

ans: SF ⇒ $(D^2 - 4D + 4)y = 0$

AE ⇒ $\lambda^2 - 4\lambda + 4 = 0$

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4\times 1\times 4}}{2\times 1}$$

$$= \frac{4 \pm \sqrt{16-16}}{2} = \frac{4}{2} \;\&\; \frac{4}{2} = \underline{2 \;\&\; 2}$$

C.F i.e $y = (c_1 + c_2 x)e^{2x}$

$$P.I. = \frac{1}{F(D)}\, g(x)$$

$$= \frac{1}{D^2-4D+4} \sin 2x.$$

$D^2 = -(a^2) = -2^2 = \underline{-4}$

$$= \frac{1}{-4-4D+4} \sin 2x.$$

$\int \sin 2x = -\frac{\cos 2x}{2}$

$$= \frac{1}{-4D} \sin 2x$$

$$= -\frac{1}{4} \cdot \frac{1}{D} \sin 2x$$

$$= -\frac{1}{4} \times -\frac{\cos 2x}{2} = \underline{\frac{1}{8} \times \cos 2x}$$

ie, $\overset{C.S}{y} = C.F + P.I$

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{8}\cos 2x$$

i.e, $y(x) = (c_1 + c_2 x) e^{2x} + \frac{1}{8} \cos 2x$ $\quad - 8,$

$c_2(x e^{2x})$ $\qquad$ $Dy = y'(0) =$

$y\ \cancel{(x)} = C_1 + \frac{1}{8}$

$y(0) = \frac{1}{8} \Rightarrow C_1 + \frac{1}{8} = \frac{1}{8} \Rightarrow C_1 = \underline{\underline{0}}.$

$Dy = y'(0)$

$y'(x) = 2C_1 e^{2x} - C_2(x \times 2e^{2x} + e^{2x} \times 1) + \frac{-2 \times \sin 2}{\cancel{8}}$

$y'(x) = 2C_1 e^{2x} - 2C_2 x e^{2x} + C_2 e^{2x} - \frac{2 \sin 2x}{8}$

$y'(0) = 2C_1 + C_2 \Rightarrow 2C_1 + C_2 = 4$

$2 \times 0 + C_2 = 4 \Rightarrow \underline{\underline{C_2 = 4}}$

∴ The particular solution i.e y_p i.e

$y = 4x\underline{e}^{2x} + \frac{1}{8} \cos 2x$

when:
$x = \frac{\pi}{4}$

$y = 4 \times \frac{\pi}{4} e^{2 \times \frac{\pi}{4}} + \frac{1}{8} \cos 2 \times \frac{\pi}{4}$

$y = \pi e^{\pi/2} + \frac{1}{8} \cos \frac{\pi}{2}$

$y = \underline{\underline{\pi e}}^{\pi/2}$

10) Solve $\frac{d^3y}{dx^3} + y = \cos 2x$, Find $\frac{dy}{dx}$.

ans: SF ⇒ $(D^3 + 1) y = 0$

AE ⇒ $\lambda^3 + 1 = 0$

$(a+b)^3 = (a+b)(a^2 - ab + b^2)$

$(\lambda+1)\quad(\lambda^2-\lambda+1)$

roots, $\lambda=-1$

$\lambda^2-\lambda+1=0$

$$\lambda = \frac{1 \pm \sqrt{(-1)^2 + -2\times1\times1}}{2\times1} = \frac{1\pm\sqrt{-3}}{2}$$

$$= \frac{1}{2} \pm \frac{\sqrt{3}}{2}i \quad \left(\alpha=\frac{1}{2},\ \beta=\frac{\sqrt{3}}{2}\right)$$

$$C.F = c_1 e^{-x} + \underline{e^{1/2\,x}}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$P.I = \frac{1}{F(D)}\,g(x)$$

$$= \frac{1}{D^3+1}\cos 2x. \qquad (D^2=-(a^2)=-(2^2)=-\underline{4}$$

$$= \frac{1}{D\{D^2+1\}}\cos 2x$$

$$= \frac{1}{-4D+1}\cos 2x.$$

$$= \frac{1}{1-4D.}\cos 2x.$$

$$= \frac{1+4D}{(1-4D)(1+4D)}\cos 2x$$

$$= \frac{1+4D}{1-16D^2}\cos 2x \qquad (D^2=-(a^2)=-(2^2)=-\underline{4}$$

$$= \frac{1+4D}{1+16\times-4}\cos 2x \qquad \frac{16\times4}{64}$$

$$= \frac{(1+4D)}{1+64}\left(\overset{\cos}{\sin}\, 2x\right)$$

$$= \frac{1}{65}\left(\overset{\cos}{\sin}\, 2x + 4D\,\overset{\cos}{\sin}\, 2x\right)$$

$$= \frac{1}{65}\left(\overset{\cos}{\sin}\, 2x + 4\times 2\,\overset{-\sin}{\cos}\, 2x\right)$$

$$= \frac{1}{65}\left(\overset{\cos}{\sin}\, 2x - 8\,\overset{\sin}{\cos}\, 2x\right) = \frac{1}{65}\left(\cos 2x - 8\sin 2x\right.$$

C.S = CF + P.I

i.e, G.S $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right)$ ① ans.

$$+\left(\frac{1}{65}\,\overset{\cos}{\sin}\, 2x - 8\sin 2x\right)$$

~~ie Ans, y =~~

$$\frac{dy}{dx}y = c_1 e^{-x} + c_3\left(e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x\right) + c_4\sin\frac{\sqrt{3}}{2}x$$

$$+ \frac{1}{65}(\cos 2x - 8\sin 2x)$$

$$\frac{dy}{dx} = -c_1 e^{-x} + c_3\left(e^{\frac{1}{2}x}\times\frac{\sqrt{3}}{2}\times -\sin\frac{\sqrt{3}}{2}x + \cos\frac{\sqrt{3}}{2}x\times\frac{1}{2}e^{\frac{1}{2}x}\right)$$

$$+ c_4\, e \times\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}x + \frac{1}{65}\times 2\times -\sin 2x - \frac{8}{65}\times 2\cos$$

$$\frac{dy}{dx} = -c_1 e^{-x} - \frac{\sqrt{3}}{2}c_3 e^{\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x + \frac{1}{2}c_3 e^{\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x$$

$$+ \frac{\sqrt{3}}{2}c_4\cos\frac{\sqrt{3}}{2}x - \frac{2}{65}\sin 2x - \frac{16}{65}\cos 2x$$

Case-4

when $g(x)$ of the form, $e^{ax} V_1(x)$

$$P \cdot I = \frac{1}{F(D)} g(x)$$

$$P \cdot I = \frac{1}{F(D)} \left[e^{ax} \cdot V_1(x) \right]$$

$$P \cdot I = e^{ax} \cdot \frac{1}{F(D+a)} V_1(x)$$

① Solve $(D^2 - 2D + 5)y = e^{2x} \sin x$.

ans: $SF \Rightarrow (D^2 - 2D + 5)y = 0$ $\quad D+2$

$A \cdot E \Rightarrow \lambda^2 - 2\lambda + 5 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 5}}{2} = \frac{2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$(\alpha = 1, \beta = 2)$

$$C \cdot F = e^{x} (c_1 \cos 2x + c_2 \sin 2x)$$

$$P \cdot I = \frac{1}{F(D)} g(x)$$

$$= \frac{1}{D^2 - 2D + 5} (e^{2x} \cdot \sin x)$$

$(D+a) = (D+2)$

$$= e^{2x} \frac{1}{(D+2)^2 - 2(D+2) + 5} (\sin x)$$

$$= e^{2x} \frac{1}{D^2 + 4D + 4 - 2D - 4 + 5} (\sin x)$$

$D^2 = -a^2 = -1$

$$= e^{2x} \frac{1}{D^2 + 2D + 5} (\sin x)$$

Replacing, —

$(D^2 = -(a^2) = -(1^2) = -1$

$$= e^{2x} \frac{1}{-1 + 2D + 5} (\sin x)$$

$$= e^{2x} \frac{1}{2D + 4} \sin x .$$

$$= e^{2x} \frac{2D - 4}{(2D+4)(2D-4)} (\sin x) \qquad (2D)^2 - (4)^2$$

$$4D^2 - 16$$

$$= e^{2x} . \frac{(2D - 4)}{4D^2 - 16} \sin x$$

$$= e^{2x} \frac{2D \sin x - 4 \sin x}{4D^2 - 16} \qquad (D^2 = -a^2 = -1)$$

$$= e^{2x} \frac{2 \times \cos x - 4 \sin x}{-4 - 16}$$

$$= \frac{e^{2x}}{-20} (2 \cos x - 4 \sin x)$$

$\therefore$ The C.S = C.F + P.I

i.e $y = e^{x} \{ C_1 \cos 2x + C_2 \sin 2x) - \frac{e^{2x}}{20} [2\cos x - 4\sin$

② Solve $(D^2-7D-18)y = x^2 \cdot e^{4x+3}$

ans; $SF \Rightarrow (D^2-7D-18)y = 0$

$AE \Rightarrow \lambda^2 - 7\lambda - 18 = 0$

$$\lambda = \frac{7 \pm \sqrt{(-7)^2 - 4\times 1\times -18}}{2\times 1} = \frac{7 \pm \sqrt{49+72}}{2}$$

$$= \frac{7 \pm \sqrt{121}}{2}$$

$$= \frac{7 \pm 11}{2} = \frac{7+11}{2} \ \& \ \frac{7-11}{2}$$

$$= \frac{18}{2} \ \& \ \frac{-4}{2}$$

$$\lambda = \underline{\underline{9}} \ \& \ \underline{\underline{-2}}$$

$$C.F = c_1 e^{-2x} + c_2 e^{9x}$$

$$P\ I = \frac{1}{F(D)} g(x).$$

$$P \cdot I = \frac{1}{D^2-7D-18} x^2 \cdot e^{4x+3}$$

$$= e^{4x+3} \frac{1}{D^2-7D-18} x^2 \qquad D+a = D+4$$

$$= e^{4x+3} \frac{1}{(D+4)^2 - 7(D+4) - 18} x^2$$

$$= e^{4x+3} \frac{1}{D^2+8D+16-7D-28-18} x^2$$

$$e^{4x+3} \quad \frac{1}{} \quad x^2$$

$$= e^{4x+3} \frac{1}{-30\left(1-\left(\frac{D}{30}+\frac{D^2}{30}\right)\right)} x^2$$

$$\boxed{(1-x)^{-1} = 1+x+x^2+\ldots}$$

$$x = \frac{D}{30} + \frac{D^2}{30}$$ an

$$= -\frac{e^{4x+3}}{30} \left[1-\left(\frac{D}{30}+\frac{D^2}{30}\right)\right]^{-1} x^2$$

$$= -\frac{e^{4x+3}}{30} \left[1+\left(\frac{D}{30}+\frac{D^2}{30}\right)+\left(\frac{D}{30}+\frac{D^2}{30}\right)^2\right] x^2$$

$(a+b^2) = \left(\frac{D}{30}\right)^2 + 2\times\frac{D^3}{?}$ (power, 3, ie cancel

$$= -\frac{e^{4x+3}}{30} \left[1+\frac{D}{30}+\frac{D^2}{30}+\frac{D^2}{900}\right] x^2$$

D=

$$= -\frac{e^{4x+3}}{30} \left[x^2 + \frac{1}{30} D(x^2) + \frac{1}{30} D^2(x^2) + \frac{1}{900} D^2(x^2)\right]$$

$$= -\frac{e^{4x+3}}{30} \left[x^2 + \frac{2x}{30} + \frac{2}{30} + \frac{2}{900}\right]$$

$$= -\frac{e^{4x+3}}{30} \left[x^2 + \frac{\cancel{2}x}{\cancel{30}15} + \frac{31}{450}\right]$$

G.S = C.F + P.I

ie, $y = C_1 e^{-2x} + C_2 e^{9x} - \frac{e^{4x+3}}{30}\left[x^2 + \frac{x}{15} + \frac{31}{46}\right.$

9/2/16

case 5

when $g(x)$ is of the form, $g(x) = x\, V_1(x)$

[illegible]
$x \cdot \sin$

$$P \cdot I = \frac{1}{F(D)}\, x \cdot V(x)$$

$$= x \cdot \frac{1}{F(D)}\, V - \frac{F'(D)}{(F(D))^2} \ast V$$

① solve $(D^2 - 2D + 1)y = x \sin x$

$GF \Rightarrow (D^2 - 2D + 1)y = 0$

$AE \Rightarrow \lambda^2 - 2\lambda + 1 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 1}}{2 \times 1}$$

$$= \frac{2 \pm}{2} = \underline{\underline{1 \ \& \ 1}}$$

$$CF = \underline{\underline{(C_1 + C_2 x)}}\, e^{x}$$

$$P \cdot I = \frac{1}{F(D)} \cdot g(x)$$

$$= \frac{1}{D^2 - 2D + 1}\, x \cdot \sin x \cdot$$

$$P \cdot I = \frac{1}{F(D)}\, x \cdot V(x)$$

$$P \cdot I = x\, \frac{1}{F(D)}\, V - \frac{F'(D)}{\{F(D)\}^2}\, V$$

$$= x\, \frac{1}{D^2 - 2D + 1} \cdot \sin x - \frac{(2D - 2)}{(D^2 - 2D + 1)^2} \sin x \cdot$$

both are in case2

Replacing $D^2 = -a^2 = -1^2 = \underline{\underline{-1}}$

$$= \frac{1}{\quad} \sin x - \frac{(2D-2)}{\quad} \sin x$$

$$= x\cdot\left(\frac{-1}{2}\right)\cdot\frac{1}{D}\sin x - \frac{1}{2}(D-1)\cdot\frac{1}{D^2}\sin x.$$

Replace $D^2 = -a^2 = -1$

$$= -\frac{1}{2}x \cdot -\cos x - \frac{1}{2}(D-1)\cdot\frac{1}{-1}\sin x$$

$$= \frac{x\cos x}{2} + \underline{\frac{1}{2}}(D-1)\sin x$$

~~cas ix~y=~~

$$= \frac{x\cos x}{2} + \frac{1}{2} D\sin x - \sin x$$

$$= \frac{x\cos x}{2} + \underline{\underline{\frac{1}{2}(\cos x - \sin x)}}$$

c.s ie

$$y = (c_1 + c_2 x)e^x + \underline{\underline{\frac{x\cos x}{2} + \frac{1}{2}}}(\cos x - \sin x)$$

② Solve $(D^4 + 2D^2 + 1)y = x^2 \cdot \cos x$. (case 5)

$(D^2+1)^2$; $x \cdot V$

SF ⇒ $(D^4 + 2D^2 + 1)y = 0$

AE ⇒ $(\lambda^4 + 2\lambda^2 + 1) = 0$

$$(\lambda^2 + 1^2)^2 \Rightarrow \lambda^4 + 1^4 = 0$$

$$\lambda^4 = -1^4$$

$$\therefore \lambda = \underline{\underline{\pm i}},\ \underline{\underline{\pm i}} \quad (\alpha = 0, \beta = 1)$$

$$CF = e^{0x}(C_1\cos x + \underline{\underline{C_2\sin x}}) + e^{0x}(C_3\cos x + C_4\sin x)$$

$$P.I = \frac{1}{F(D)} g(x)$$

$$= \frac{1}{(D^2+1)^2}(x^2\cos x)$$

$$\cancel{\frac{1}{F(D)} x^{a} V(x) = x\frac{1}{F(D)} V - \frac{F'(D)}{(F(D))^2} V}$$

$$= \frac{1}{(D^2+1)^2} \text{ R.P of } x^2e^{ix}$$

(Applying case 4)

$\frac{1}{D^2}$ $\qquad D+a = D+i$

$$x^2e^{ix} = x^2(\cos x + i\sin x)$$
$$= x^2\cos x + i x^2 \sin x.$$

- $x^2\cos x$ = Real part of x^2e^{ix} (R.P)
- $x^2\sin x$ = Imaginary part of x^2e^{ix} (I.P)

$$= \text{R.P of } e^{ix} \frac{1}{[(D+i)^2+1]^2} x^2$$

$$(1+x)^{-n} = 1 - \frac{n}{1i}x + \frac{n(n+1)}{2i}x^2 - \frac{n(n+1)(n+2)}{3i}x^3$$

$$= \text{R.P of } e^{ix} \frac{1}{(D^2+2Di+i^2+1)^2} x^2$$

$$= \text{R.P of } e^{ix} \frac{1}{(D^2+2Di-\cancel{1}+\cancel{1})^2} x^2$$

$$= \text{R.P of } e^{ix} \frac{1}{(D^2+2Di)^2} x^2 \quad \underline{\text{case 3}}$$

$D^4 + 4i^2D^2$

$$(1-x)^{n} = 1 - \frac{n}{1i}x + \frac{n(n+1)}{2i}x^2 - \frac{n(n+1)(n+2)}{3!}x^3$$

$$= \text{R.P of } e^{ix} \frac{1}{4i^2D^2\left[1+\frac{D^2}{2iD}\right]^2} x^2$$

$(i^2=-1)$ $\quad D^8$

$$= \text{R.P of } e^{ix} \frac{1}{-4D^2}\left[1+\frac{D^2}{2iD}\right]^{-2} \times x^2$$

$$\left(1+\frac{D^2}{2iD}\right)^{-2} = 1 - \frac{2D^2}{2iD} + \frac{2(2+1)}{2i} \times \left(\frac{D^2}{2iD}\right)^2$$

$$= \text{R.P of } e^{ix} \frac{1}{-4D^2}\left(1 - \frac{2D^2}{2iD} + \frac{6}{2!}\left(\frac{D^2}{2iD}\right)^2\right] \times x^2$$

$\frac{1}{i} \times D = -iD + 3 \times \frac{1}{1} \times \frac{D^4}{4i^2D^2}$

$$\boxed{\frac{1}{i} = \frac{i}{i^2} = -}$$

$$= \text{R.P of } e^{ix} \frac{1}{-4D^2}\left[1 + iD \pm \frac{3}{4}D^2\right] x2 \qquad \underline{\underline{2! = 2}} \qquad \frac{1}{i^3} = \frac{i}{i^4}$$

$$= \text{R.P of } e^{ix}\left(\frac{-1}{4}\right)\left(\frac{i}{D^2}\right)\left[x^2 + iD(x^2) - \frac{3}{4}D^2(x^2)\right] \qquad \frac{2x}{2}$$

$$= \text{R P of } e^{ix}\left(-\frac{1}{4}\right)\left(\frac{1}{D^2}\right)\left[x^2+2ix - \frac{3}{4}\times 2\right]$$

$$= R\cdot P \text{ of } e^{ix}\left(\frac{-1}{4}\right)\left(\frac{1}{D}\right)\left(\frac{1}{D}\right)\left[x^2+2ix-\frac{3}{2}\right]$$

$$= R\cdot P \text{ of } e^{ix}\left(\frac{-1}{4}\right)\frac{1}{D}\int x^2+2ix-\frac{3}{2}\,dx.$$

$$= R\cdot P \text{ of } e^{ix}\left(\frac{1}{4}\right)\frac{1}{D}\left[\frac{x^3}{3}+2i\frac{x^2}{2}-\frac{3}{2}x\right]$$

$$= R\cdot P \text{ of } e^{ix}\left(\frac{-1}{4}\right)\int\frac{x^3}{3}+i\,x^2-\frac{3}{2}x\,dx.$$

$$= R\cdot P \text{ of } e^{ix}\left(\frac{-1}{4}\right)\left[\frac{x^4}{12}+i\frac{x^3}{3}-\frac{3x^2}{4}\right]d$$

$$= R\cdot P \text{ of } [\cos x+i\sin x]\left[\frac{-1}{4}\right]\left[\frac{x^4}{12}+i\frac{x^3}{3}-\frac{3x^2}{4}\right]$$

Realpart + i(imaginary part

$$= R\cdot P \text{ of}\Big(-\frac{x^4\cos x}{4\times 12}+-\cos x\frac{i\,x^3}{4\times 3}+-3\frac{x^2\cos x}{-4\times 4}+$$

$$+-i\frac{x^4\,i\sin x}{12\times 4}+\frac{i^2 x^3\times i\sin x}{-4\times 3}+-\frac{3x^2 i\sin x}{-4\times 4}\Big)$$

$$= RP \text{ of}\Big(-\frac{x^4\times i\sin x}{48}-\frac{x^3\times i\cos x}{12}+3\frac{x^2\cos x}{16}$$

$$-\frac{x^4\times i\sin x}{48}-\frac{\overset{(-1)}{i^2}x^3\times\sin x}{12}+\frac{3x^2\times i\sin x}{16}\Big).$$

$$= RP \text{ of }\Big[-x^4 i\sin x - x^3\times i\cos x + 3\,x^2\cos x$$

$$[illegible] \cos x \; \frac{x^3}{12} - \frac{3x^2}{4} + i \sin x \; \frac{x^3}{3}$$

$$[illegible] \cos x \left(\frac{x^4 - 9x^2}{12}\right) + i\,\frac{x^3 \sin x}{3}$$

$$[illegible] \cos x \; \frac{x^4 - 9x^2}{48} - \frac{i\,x^3 \sin x}{12}$$

$$P.I = \frac{1}{48}\left[4x^3 \sin x - \left(x^4 - 9x^2\right)\cos x\right]$$

$$G.S = c_1 \cos x + c_2 \sin x + c_3 \cos x + c_4 \sin x + \frac{1}{48}\left(4x^3 \sin x - \left(x^4 - 9x^2\right)\cos x\right)$$

Method of variation of parameter

For Finding P.I of second order Linear ODE, using method of variation of parameter

This method is quite general and applicable ~~of~~ only for the ODE of the form

$$y'' + p(x)y' + q(x)y = g(x)$$

Let, the C.F of the above equations is, $y = c_1 y_1 + c_2 y_2$, where c_1, c_2 are arbitary constants and y_1, y_2 are independent solutions. Then the P.I of the above equation is of the form i.e, ~~$P.I = y_1 \int y_2$~~ $\left(W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \right.$

$$\boxed{P.I = -y_1 \int \frac{y_2 g}{W}\,dx + y_2 \int \frac{y_1 g}{W}\,dx.}$$

Cauchy's Homogeneious Linear D.E. This Homogeneous Linear equation can be reduced to linear equation with constant coefficient by changing the independent variable x to z.

* $\boxed{x = e^z}$ $\boxed{z = \log x}$ $\frac{dz}{dx} = \frac{1}{x}$

* $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$ $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$

* $x \frac{dy}{dx} = \frac{dy}{dz}$ $x\frac{dy}{dx} = D_1 y.$

similarly $x^2 \frac{d^2y}{dx^2} = D_1 (D_1 - 1)y$ and

$$x^3 \frac{d^3y}{dx^3} = D_1 (D_1-1)(D_1-2)y \dots\dots$$

$$x^n \frac{d^ny}{dx^n} = D_1 (D_1-1) \dots D_1-(n+1)y$$

$(D_1 - n+1)$

Substituting these terms in the given D.E, it will be definitely change to ~~the~~ a LDE with constant coefficients and we can solve the problem by ~~this~~ the method discussed above.

① Solve $x^2 \frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x,$

ans: Clearly this ~~is~~ (the give eqn) is a H.L.D.E, and this can be reduced to the Ordinary D.E with constant coefficient by the substitution

$\Downarrow$

$\boxed{x\frac{dy}{dx} = D_1 y \quad x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y}$

$\boxed{x = e^z}$

ie $(D_1(D_1-1) - 4D_1 + 6)y = e^z$

$(D_1^2 - D_1 - 4D_1 + 6)y = e^z$

$\Rightarrow (D_1^2 - 5D_1 + 6)y = e^z$

AE → ~~$m_1^2 - 5m + 6$~~ $= 0$

$\lambda_1^2 - 5\lambda + 6 = 0$

$$\lambda = \frac{5 \pm \sqrt{25 - 4 \times 1 \times 6}}{2 \times 1} = \frac{5 \pm \sqrt{1}}{2}$$

$$= \frac{5 \pm 1}{2} = \frac{5+1}{2} \ \& \ \frac{5-1}{2}$$

$$= \frac{6}{2} \ \& \ \frac{4}{2}$$

$$m = 3 \ \& \ 2$$

C.F i.e

$$y = C_1 e^{2z} + C_2 e^{3z}$$

$$y = C_1 e^{2} + C_2 e^{3} \qquad (e^{z} = x)$$

$$P.I = \frac{1}{F(D)} x$$

$$= \frac{1}{D_1^2 - 5D_1 + 6} e^{z}$$

$$(D = a = 1)$$

$$= \frac{1}{1^2 - 5 + 6} e^{z}$$

$$= \frac{1}{2} e^{z}$$ ~~case of failure~~

$$= \frac{1}{2} e^{z} \qquad = \frac{1}{2} x.$$

$$\therefore \text{CF} = y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{2} e^{z}$$

$$y = C_1 x^2 + C_2 x^3 + \frac{1}{2} x$$

② Solve $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + y = (\log x)(\sin \log x + 1$

ans: Definitely the given eqn, is a H.L.D.E.

This can be Reduced to the Ordinary L.D.E. with onstant coefficient by the substitution

$D_1(D_1-1) - 3(D_1 + 1)y = z\left[\sin z+1\right]$

$D_1(D_1-1) - 3D_1 + 1)y = z\sin z + z$

$x\frac{dy}{dx} = D_1 y$

$x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y$

$x = e^z$

$\log x = z$

$\sin\log x = \sin z$

he S.F.)

$D_1^2 - D_1 - 3D_1 + 1)y = z\sin z + z$

$(D_1^2 - 4D_1 + 1)y = 0$

$\Rightarrow \lambda_1^2 - 4\lambda_1 + 1 = 0$

$\lambda = \frac{4 \pm \sqrt{16-4\times1\times1}}{2\times1}$

$2\,|\,12$

$2\,|\,6$

3

$2\sqrt{3}$

$\lambda = \frac{4 \pm 2\sqrt{3}}{2} = \frac{2\pm\sqrt{3}}{\cancel{2}} = 2 \pm \frac{2\sqrt{3}}{2} = \underline{\underline{2\pm\sqrt{3}}}$

$\underline{\underline{CF = c_1 e^{(2+\sqrt{3})z} + c_2 e^{(2-\sqrt{3})z}}}$

$P.I = \frac{1}{F(D)}\,g(x)$

(case 5) $(z' = x^m)$

$= \frac{1}{D_1^2 - 4D_1 + 1}\left[\underbrace{z\sin z}_{P.I_1} + \underbrace{z}_{P.I_1}\right]$

$P.I_1 = \frac{1}{D_1^2-4D_1+1}\,z\sin z$

$z \cdot V$

$P.I = z\frac{1}{F(D)}V - \frac{F'(D)}{(F(D))^2}V$

$= z\frac{1}{D_1^2-4D_1+1}\sin z - \frac{(2D_1-4)}{(D_1^2-4D_1+1)^2}\sin z$

$(D^2 = -a^2 = -1^2 = \underline{-1})$

$= z\frac{1}{-1-4D+1}\sin z - \frac{(2D_1-4)}{(-1-4D+1)^2}\sin z$

③ $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \log x$.

Clearly this is Cauchy's equation. This can be reduced to the ordinary linear D.E by the substitution $\underline{x = e^z}$ or $\underline{z = \log x}$,

$x\frac{dy}{dx} = D_1 y$, $x^2\frac{d^2y}{dx^2} = D_1(D_1-1)y$

$\Rightarrow$

$(D_1(D_1-1) - D_1 + 1)y = z$

$(D_1^2 - D_1 - D_1 + 1)y = z$

$\Rightarrow (D_1^2 - 2D_1 + 1)y = 0$.

AE $\Rightarrow$ $\lambda_1^2 - 2\lambda_1 + 1 = 0$

$$\lambda = \frac{2 \pm \sqrt{4 - 4\times1\times1}}{2\times1} = \frac{2\pm}{2} \Rightarrow \underline{1} \,\&\, \underline{1}$$

$\lambda = 1, 1$.

CF $= (c_1 + c_2 x)e^x$. $(x \to z)$

CF i.e $y = (C_1 + C_2 z)e^z$

(case 3)

$[1 + (D_1^2 - 2D_1)]^{-1}$

$$P.I = \frac{1}{D_1^2 - 2D_1 + 1} z^1$$

$$= \frac{1}{(D^2 - 2D_1) + 1} z^1$$

$D^2 = a^2 = 1$

$$= [1 + (D_1^2 - 2D_1)]^{-1} z^1$$

$$= (1 + 2D_1)z^1$$

$$= z + 2D_1(z)$$

$$= \underline{z + 2}$$

C.S. i.e

$y = (C_1 + C_2 z)e^z + z + 2$

$\underline{y = (C_1 + C_2 \log x)e^{1}x + \log x + 2}$

④ Solve $x^2 \frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\log x(\log \sin x)+1}{x}.$
Q.

Legender's Linear D. E (s)

An equation of the form ~~a+bx)~~

$$(a+bx)^n \frac{d^ny}{dx^n} + K_1(a+bx)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} +$$

$$K_2(a+bx)^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + K_n y = g(x$$

where all case are constants and $g(x)$ is function of 'x' alone. For solving the abov ODE, we have to change this into Ordin Linear D·E by the substitution of ~~(a+bx)~~

$$\boxed{(a+bx) = e^z \quad \text{i.e,} \quad z = \log(a+bx)}$$

i.e $\frac{dz}{dx} = \frac{1}{a+bx}\cdot b = \frac{b}{a+bx}$

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{dy}{dx}\ \frac{b}{a+bx}.$$

$$(a+bx)\frac{dy}{dx} = b\cdot\frac{dy}{dx}$$

$$= b\frac{d}{dz}(y) = \underline{b\cdot D_1 y}$$

Similarly, $(a+bx)^2 = \frac{d^2y}{dx^2} = b^2 D_1(D_1-1)y$

$$(a+bx)^3 = \frac{d^3y}{dx^3} = b^3 D_1(D_1-1)(D_1-2)y$$

and so on. and can be solved by the method of discussed above

① Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 2 \sin \log(1+x)$

Ans: Clearly the equation is Legender's Linear equations and this can be converted to the ordinary Linear D.E with constant coefficients by the substitution of $(a+bx) = e^z$, $\underline{z = \log(a+bx)}$

ie $(1+x)\frac{dy}{dx} = 1 D_1 y$

$$(1+x)^2 \frac{d^2y}{dx^2} = 1^2 D_1(D_1-1)y$$

$$[1^2 D_1(D_1-1) + D_1 + 1]y = 2 \sin z$$

SF ⇒ $(D_1^2 + 1)y = 2 \sin z$

AE ⇒ $(\lambda^2 + 1) = 0$

$$\lambda^2 = \sqrt{-1} \Rightarrow \underline{\underline{\lambda = \pm i}} \quad (\alpha = 0, \beta = 1)$$

C.F ie, $y = e^{0x}(c_1 \cos x + c_2 \sin z)$

ie $y = (c_1 \cos z + c_2 \sin z)$

$$P.I = \frac{1}{1+\underline{D_1^2}} 2 \sin z$$

$$= 2 \frac{1}{1+D_1^2} \sin z$$

$D_1^2 = -a^2 = -1^2 = \underline{\underline{-1}}$

$= 2 \frac{1}{1-1} \sin z = \underline{\underline{\frac{2}{0} \sin z}}$ the case of failure

(differentiating the denominator & multiply the numerator with z)

$$P.I = 2z \cdot \frac{1}{2D_1} \sin z$$

$$= \underline{2z} \frac{1}{D_1} \sin z$$

cs ie $y = CF + PI$

$y = c_1 \cos z + c_2 \sin z - z \cos z$

$\therefore y = c_1 \cos(\log(\overset{1+x}{a+bx})) + c_2 \sin[\log(\overset{1+x}{a+bx})]$
$- \log(\overset{1+x}{a+bx}) \cos[\log(\overset{1+x}{a+bx})]$

ontinue:
Cauchy's LDE)

② ∴ C.S is, $y = c_1 e^{(2+\frac{1}{\sqrt{3}})\log x} + c_2 e^{(2-\sqrt{3})\log x} + \frac{1}{8}(2z\cos z$

$\cos z - 2\sin z) + z + 4$

$y = c_1 e^{(2+\sqrt{3})\log x} + c_2 e^{(2-\sqrt{3})\log x} + \frac{1}{8}\left[2\log x \cdot\right.$

$\cos(\log x) + \cos\log x - 2\sin(\log x)\left.\right]$;

$\log x + 4$

egenders

② $(3x+2)^2\frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2 + x + 1.$

ans: Clearly this is a Legender Linear equatio having the standard form

$$(3x+2)^2\frac{d^2y}{dx^2} + 5(3x+2)\frac{dy}{dx} - 3y = x^2 + x + 1$$

$$(ax+b)^2\frac{d^2y}{dx^2} + 5(ax+b)\frac{dy}{dx} - 3y = g(x).$$

and it can be solved by applying the following rearrangement. or substitution.

$e^z = 3x+2$, $z = \log(3x+2)$

$\Rightarrow x = \frac{e^z-2}{3}$, & ~~$x\frac{dy}{dx} =$~~

- $x\frac{dy}{de} = a\,D_1 y = 3\,D_1 y$
- $x^2\frac{d^2y}{dx^2} = a^2 D_1(D_1-1)\,y = 3^2 D_1(D_1-1)y$

∴ The equation becomes,

$$\left[3^2 \times D_1(D_1-1) + 5\times 3D_1 - 3\right] y = \left(e^z-2\right)^2 + e^z - 2 + 1$$

$(a-b)^2$

$$[9(D_1^2 - D_1) + 15D_1 - 3]y = \frac{e^{2z} - 4}{9 \times 3} + \frac{e^z - 2}{3} + 1$$

$$(9D_1^2 - 9D_1 + 15D_1 - 3)y = \frac{(e^{2z} - 4)3 + (e^z - 2)9}{27} + 1.$$

~~$(9D_1^2 - 4D_1 - 3)y$~~

$$e^{2z} + e^z - 4 \cdot 2 + 1$$

$$(9D_1^2 + 6D_1 - 3)y = \frac{e^{2z} - e^z + 7}{9} \qquad \left(\frac{e^{2z} - 4e^z + 4}{3^2}\right) + \frac{e^z - 3}{3}$$

Now the A·E is

$$9\lambda^2 + 6\lambda - 3 = 0$$

$$\lambda = \frac{-6 \pm \sqrt{36 - 4 \times 9 \times -3}}{2 \times 9} = \frac{-6 \pm \sqrt{36 + 108}}{18}$$

$$= \frac{-6 \pm \sqrt{144}}{18} = \frac{-6 \pm 12}{18}$$

$$= \frac{-6 + 12}{18} \;\&\; \frac{-6 - 12}{18}$$

$$= \frac{6}{18} \;\&\; -\frac{18}{18}$$

$$= \frac{1}{3} \;\&\; -1$$

∴ CF i.e

$$y = C_1 e^{-z} + C_2 e^{\frac{1}{3}z}$$

$$P \cdot I = \frac{1}{F(D)} g(x)$$

$$P \cdot I = \frac{1}{9D_1^2 + 6D_1 - 3} \times \frac{e^{2z} - e^z + 7}{9}$$

$$= \frac{1}{9D_1^2 + 6D_1 - 3} \Big(\underbrace{\frac{e^{2z}}{9}}_{P \cdot I_1} - \underbrace{\frac{e^z}{9}}_{P \cdot I_2} + \underbrace{\frac{7}{9}e^{0z}}_{P \cdot I_3} \Big) \cdot x^0$$

$$P \cdot I_1 = \frac{1}{9D_1^2 + 6D_1 - 3} \frac{e^{2z}}{9}$$

$$= \frac{1}{9} \times \frac{1}{9D_1^2 + 6D_1 - 3} e^{2z}$$

$$(D_1 = a = 2)$$

$$P \cdot I_1 = \frac{1}{9} \frac{1}{9 \times 2^2 + 6 \times 2 - 3} e^{2z}$$

$$= \frac{1}{9} \times \frac{1}{36 + 18 - 3} e^{2z}$$

$$= \underline{\underline{\frac{e^{2z}}{405}}}$$

$$P \cdot I_2 = \left(-\frac{1}{9}\right) \cdot \frac{1}{9D_1^2 + 6D_1 - 3} e^{z}$$

$$(D_1 = a = 1)$$

$$= \left(\frac{-1}{9}\right) \times \frac{1}{9 \times 1^2 + 6 \times 1 - 3} e^{z}$$

$$= \left(\frac{-1}{9}\right) \times \frac{1}{12} e^{z} = \underline{\underline{-\frac{e^{z}}{108}}}$$

$$P \cdot I_3 = \frac{7}{9} \frac{1}{9D_1^2 + 6D_1 - 3} e^{0z}$$

$$(D_1 = a = 0$$

$$= \frac{7}{9} \frac{1}{9 \times 0 + 6 \times 0 - 3} e^{0z}$$

$$= \underline{\underline{\frac{-7}{27}}}$$

$$\therefore \text{C.S ie } y = c_1 e^{-z} + c_2 e^{1/3 z} + \frac{e^{2z}}{405} - \frac{e^{z}}{108} - \frac{7}{6}$$

$$\therefore z = \log(3x+2)$$

$$y = c_1 e^{-\log(3x+2)} + c_2 e^{1/3 \log(3x+2)} + \frac{e^{2\log(3x+2)}}{405} - \frac{e^{\log(3x+2)}}{108}$$

Non-periodic functions

The following functions are non-periodic

(i) $F(x) = k$.

(ii) $F(x) = \begin{cases} 1, & x \text{ is rational} \\ 0, & x \text{ is irrational} \end{cases}$

Orthogonality of sine and cosine functions

The collection of linearly independent functions. $\left\{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \cos\frac{3\pi x}{L}, \dots, \sin\frac{\pi x}{L}, \sin\frac{2\pi x}{L}, \sin\frac{3\pi x}{L}, \dots\right\}$ Form what is called a mutually orthogonal set of functions.

In this case of functions we say that the two functions $f(x)$ & $g(x)$ are orthogonal in an interval $[a, b]$ if $\int_a^b f(x) g(x)\,dx = 0$ where the above set of linearly independent functions are analogous to a mutually ⊥r set of vectors in vector analysis

A set of function is called Mutually orthogabal

A set of functions is orthogonal. if each distinct pair of functions is orthogonal. sine & cosine term

thus, the above collection of sine & cosine term forms a mutually orthogonal set of functions on the closed (C & C+2L) where C is any constan

1) $$\int_C^{C+2L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \\ 2L, & m = n = 0 \end{cases}$$

2) $$\int_C^{C+2L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0 \quad \forall\ m \& n$$

$$\int_0 \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0, & m \neq n \\ L, & m = n \\ 0, & m = n = 0. \end{cases}$$

Trignometric Series

A series of the form
$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos x + b_2 \sin x \ldots\ldots$

or $a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + a_3 \cos 3x + b$
$+ \ldots\ldots$

are called a trignometric series.

The above series with a desired rang of values or intervels are called Fourier Series.

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Dirichlet's conditions

That are sufficient for a function f(x) to have a fourier series represent

The directed conditions are

① f(x) is periodic and finit
② f(x) has finite number of discontinuity in any one period.
③ f(x) has at the most a finite num maxima and minima in any one

Fourier Series and Eulur's Formula

A series of the form $f(x) = \frac{a_0}{2} +$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) \quad \text{①}$$

where a_0, a_n, and b_n are Fourier coefficients

$(0 < x < 2L)$

$(c < x < c+2L)$

Euler's Formula

The coefficient a_0, a_n and b_n are determined by Euler's formula, with term by term integration and with the help of orthogonality of sine and cosine terms.

Here we are simply integrating the ① with respect to x, from 'c' to '$c+2L$'

$\sin n\pi = 0$

$-\cos n\pi$

$\cos n\pi - \cos n\pi = 0$

$$\int_c^{c+2L} f(x)\,dx = \frac{a_0}{2}\int_c^{c+2L} dx + a_n \int_c^{c+2L} \sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{L}\right) + b_n \int_c^{c+2L} \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_c^{c+2L} f(x)\,dx = \frac{a_0}{2}\left[x\right]_c^{c+2L} + a_n \times 0 + b_n \times 0.$$

$$\int_c^{c+2L} f(x)\,dx = \frac{a_0}{2} \times \left[c+2L - c\right]$$

$$= \frac{a_0}{2} \times 2L$$

$$= a_0 L$$

$$a_0 = \frac{1}{L}\int_c^{c+2L} f(x)\,dx.$$

To find ~~the~~ a_n multiply ① throughout

$$\int_{C}^{C+2L} f(x)\,\cancel{dx}\cdot\cos\frac{m\pi x}{L}\,dx = \frac{a_0}{2}\int_{C}^{C+2L} \cancel{\cos\left(\frac{n\pi x}{L}\right)}\cos\frac{m\pi x}{L}\,dx +$$

$$a_n\int_{C}^{C+2L}\sum_{n=1}^{\infty}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx + b_n\int_{C}^{C+2L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\cdot\cos\left(\frac{m\pi x}{L}\right)$$

$\boxed{m=n=1}$

$$\therefore \int_{C}^{C+2L} f(x)\,\cancel{dx}\cdot\cos\left(\frac{m\pi x}{L}\right)dx$$

$$= \frac{a_0}{2}\times 0 + a_n \times L + b_n\times 0$$

$$= a_n L$$

$$\therefore a_n = \frac{1}{L}\int_{C}^{C+2L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx. \quad (m=n=1$$

Since 'm' is kept fixed and 'n' ranges over the +ve integers, it follows from the orthogonality condition that ~~n~~ n=m for all tor

ie, $$\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\left(\cos\frac{n\pi x}{L}\right)dx.}$$

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② To find b_n

Here we are multiplying, the ① throughout by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating w. r. to 'x' from c to C+2L

ie $$\int_{C}^{C+2L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{a_0}{2}\int_{C}^{C+2L}\sin\left(\frac{m\pi x}{L}\right)dx +$$

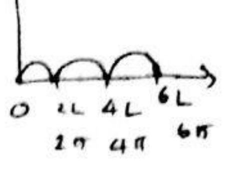
$$a_n\int_{C}^{C+2L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx + b_n\int_{C}^{C+2L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)d$$

$$\int_{c}^{c+2L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx = \frac{a_0}{2} \times 0 + a_n \times 0 + b_n \times L$$

$$\int_{c}^{c+2L} f(x) \sin\left(\frac{m\pi x}{L}\right) dx = L b_n$$, Here $m = n$ as in the case of an due to mutually orthogonality condition.

$$\boxed{\therefore b_n = \frac{1}{L} \int_{c}^{c+2L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx}$$

* The Fourier series, is a series having the periodicity 2L or 2π



$2L$
$$\boxed{\begin{array}{ll} & c < x < c+2L \\ c=0 & 0 < x < 2L \\ c=-L & -L < x < L \end{array}}$$

2π
$$\boxed{\begin{array}{ll} & c < x < c+2\pi \\ c=0, & 0 < x < 2\pi \\ c=-\pi, & -\pi < x < \pi \end{array}}$$

Particular Cases

(a) If $f(x)$ is a periodic function of period 2L defined in $-L \to 2L$ $[-L, 2L]$ which is obtained by $p = -L$ in the interval $[c, c+2L]$ in this case the Fourier series of $f(x)$ is given by

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \left(\sin \frac{n\pi x}{L}\right)}$$

and the Euler's Formula are given by

$$a_0 = \frac{1}{L}\int_{-L} f(x)\,dx.$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cancel{d}\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$

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(b) If f(x) is a periodic function with period 2π ie, $[c, c+2\pi]$ is obtained by putting $L = 2\pi$ in $[c \to c+2L]$. In this case, the Fourier series of f(x) is given by ~~f(x)~~

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx).$$

(L = π) ~~In A~~ eqn.

where

$$a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos(nx)\,dx$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin(nx)\,dx.$$

(c) If we put $c = -\pi$ in $c \to c+2\pi$ ie, then f(x) become a periodic function of period '2π' defined in $-\pi \to \pi$. Then, the fourier series is given by.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + \sum_{n=1}^{\infty} b_n\sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

Dirichlet's Theorem.

- Is $f(x)$ is continuous then it converge to $f(x)$,
- If $f(x)$ is discontinuous at a point x_0, then the Fourier series converges to the average of the right and left hand limits of the function $f(x)$, at x_0. i.e, the Fourier series converges to $\dfrac{f(x_0^+) + f(x_0^-)}{2}$

1) Find the Fourier series for the two periodic, functions

$$f(x) = \begin{cases} 0, & -1 < x < 0 \rightarrow -L < x < 0 \\ 1, & 0 < x < 1 \rightarrow 0 < x < L \end{cases}$$

$[-L, L]$

$2L = 2$

$\Rightarrow \underline{\underline{L = 1}}$

By Euler's formula, we have

$$\boxed{a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx.}$$

$$\therefore a_0 = \frac{1}{1} \int_{-1}^{1} f(x)\, dx. \overset{\text{discontinuity}}{} = 1\left[\int_{1}^{0} \overset{(0)}{\cancel{f(x)}}\, dx + \int_{0}^{1} \cancel{f(x)}(1)\, dx\right]$$

$$= 1\,[x]_0^1$$

$$= 1 - 0 = \underline{\underline{1}}. \qquad \Rightarrow \therefore \underline{\underline{a_0 = 1}}$$

$$\boxed{a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx.}$$

$$\therefore a_n = \frac{1}{1} \int_{-1}^{1} f(x) \cos\left(\frac{n\pi x}{1}\right) dx.$$

$$= 1 \left[\int_{-1}^{0} 0 \times \cos\left(\frac{n\pi x}{\cancel{2}}\right) dx + \int_{0}^{1} 1 \cos\left(\frac{n\pi x}{\cancel{L}}\right) dx.\right.$$

$\boxed{\sin n\pi = 0}$

$$= \left[\frac{\sin n\pi x}{n\pi}\right]_0^1$$

$$= \frac{1}{n\pi} \sin n\pi - \sin 0 = \frac{1}{n\pi} \sin \pi = \frac{1}{n\pi} \times 0 = \underline{\underline{0}}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx}$$

$$\therefore b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx.$$

$$= 1\int_{-1}^{0}(0)\sin(n\pi x)\,dx + \int_{0}^{1} 1\sin(n\pi x)\,dx$$

$$= \left(-\frac{\cos n\pi x}{n\pi}\right)_0^1$$

$$\boxed{\cos(n\pi) = (-}$$

$$= -\frac{1}{n\pi}\left(\cos n\pi \times 1 - \cos 0\right)$$

$$= -\frac{1}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{1-(-1)^n}{n\pi}\begin{cases}0, & n \text{ is even}\\ \frac{2}{n\pi}, & n \text{ is odd.}\end{cases}$$

- if n odd $= -1$
  i.e $1 - -1 = 2$
- if n ~~is~~ even $=$
  $(-1)^0 = 1$
  i.e $1 - 1 = 0$

$\therefore$ our Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}(0)\times\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{1}{2} + \cancel{\sum}\frac{2}{\pi}\left[\sin\frac{\pi x}{1} + \sin\frac{3\pi x}{3} + \sin\frac{5\pi x}{5} + \cdots\right]$$

(odd $= \pi, 3\pi, 5\pi$

2) Find the Fourier series the periodic function $f(x)$ with period 2π, defined as follow. ~~F(x)=0 for where~~ $F(x) = \begin{cases}0, & -\pi < x < 0\\ x, & 0 \le x \le \pi\end{cases}$, what is the sum of the series at $x = 0, \pm\pi, 4\pi, -5\pi$

ans: Here the given problem lies in the interval $-\pi \to \pi$ i.e, $\boxed{a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx}$

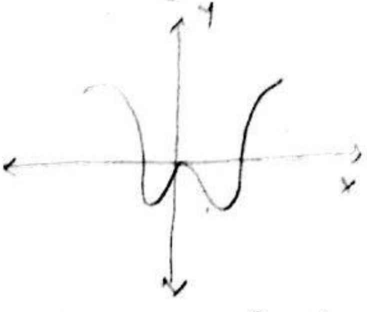
22|2|2016
Monday

Module-3 (continue)

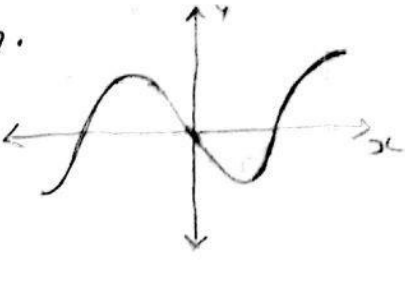
Even and odd Functions.

- A real valued function $f(x)$ is said to be even is $f(-x) = f(x)$. The graph of an even function is symmetrical about the y axis

Example :
x^2, $|x|$, $\cos x$ etc.



- A real valued function is said to be odd. if $f(-x) = -f(x)$. The graph of an odd function is symmetrical about the origin.



Example :
x, x^3, $\sin x$.

Note :

1. The product of two even functions are even.
2. The product of two odd functions are also even.
3. The product of an even and odd function is odd.

Example.

*1) If 'f' is defined by $f(x) = k.$, where 'k' is a constant is an even function.

1 $\therefore \pi + x^2$ is an even function
(i.e, x^2 is even, π is constant, so that also even.

22/2/2016 Monday

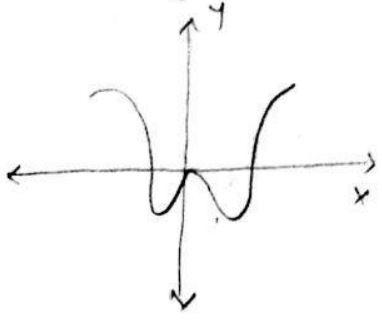
Module-3 (continue.)

Even and odd Functions.

- A real valued function $f(x)$ is said to be even is $f(-x) = f(x)$. The graph of an even function is symmetrical about the y axis

Example :

x^2, $|x|$, $\cos x$ etc.



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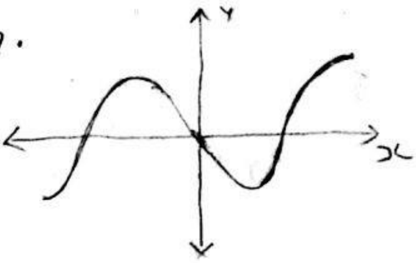

Example :

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Note :

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Example.

*1) If 'f' is defined by $f(x) = k.$, where 'k' is a constant is an even function.

1 $\therefore \pi + x^2$ is an even function

(i.e, x^2 is even, π is constant, so that also even.

2) e^{x^2}

x^2 is an even. So e^{x^2} is also an even function.

3) $x + x^2$.

x is an odd, x^2 is an even

$x + x^2$ is neither odd nor even

4) e^x & $\log x$.

e^x & $\log x$ are neither odd nor even.

If $f(x)$ is an even function, then

5) $f(x) \cdot \cos nx$ is even.

6) If $f(x)$ is an even function then $f(x) \cdot \sin nx$.

So, $f(x) \cdot \sin nx$ is odd.

7) If $f(x)$ is an odd function, then $f(x) \cdot \sin x$.

So, $f(x) \cdot \sin nx$ is an even function.

8) If $f(x)$ is an odd function, then $f(x) \cdot \cos n$

$\therefore f(x) \cdot \cos nx$ is an odd function.

Properties of even and odd functions

- If $f(x)$ is an even function, then

$$\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx.$$

- If $f(x)$ is an odd function, then

$$\int_{-c}^{c} f(x)\,dx = 0$$

* If f(x) is an even function, then [-c, c]

$$a_0 = \frac{1}{c}\int_{-c}^{c} f(x)\,dx.$$

$$= \frac{2}{c}\int_{0}^{c} f(x)\,dx.$$

[Clearly this is a Fourier cosine series]

Similarly, $$a_n = \frac{1}{c}\int_{-c}^{c} f(x)\cos\left(\frac{n\pi x}{c}\right)dx.$$

$$= \frac{2}{c}\int_{0}^{c} f(x).\cos\left(\frac{n\pi x}{c}\right)dx.$$

$$bn = \frac{1}{c}\int_{-c}^{c} f(x).\sin\left(\frac{n\pi x}{c}\right)dx.$$

$$= \underline{0}.$$

Note:
If f(x) is an even function, then a_0 and a_n will exist and bn will vanished.

Similary,
* If f(x) is on odd function in the interval [-c, c]

[clearly this is a Fourier sine series]

$$a_0 = \frac{1}{c}\int_{-c}^{c} \underset{odd}{f(x)}\,dx = \underline{\underline{0}}$$

$$a_n = \frac{1}{c}\int_{-c}^{c} \underset{odd}{f(x)}.\underset{even}{\cos\left(\frac{n\pi x}{c}\right)}\,dx = \underline{\underline{0}}$$

odd → ∫odd = 0

$$bn = \frac{1}{c}\int_{-c}^{c} \underset{odd}{f(x)}.\underset{odd}{\sin\left(\frac{n\pi x}{c}\right)}\,dx. = \frac{2}{c}\int_{0}^{c} f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

even

Note:
If f(x) is odd, then a_0 and a_n will vanished and bn will exist.

. If f(x) is even function in the interval [-c, c] then the Fourier series become ~~F(x) =~~ .

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$

· If f(x) is an odd function in the interval [-c, c
then the fourier series.

$$f(x) = \sum_{n=1}^{\infty} bn \ \sin\left(\frac{n\pi x}{c}\right)$$

Problem

1) Represent the function $F(x) = x^2$, as a ~~fou~~ Fourier series in the interval $(-\pi, \pi)$ and shown that

(1) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$

(2) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots = \frac{\pi^2}{12}$

(3) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

ans: Here the function $f(x) = x^2$ is an even function
Here 'a₀ & aₙ' will exist and 'bₙ' will vanish.

So, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left(\frac{x^3}{3}\right)_0^{\pi} = \frac{2}{\cancel{\pi}}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3} = \frac{2}{3}\pi^2$$

Similarly

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cdot\cos\left(\frac{n\pi x}{\pi}\right)dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx.$$

$\int uv\,dx = uv_1^* - u'v_2 + u'$

$$= \frac{2}{\pi}\int_0^{\pi} x^2\cdot\cos(nx)\,dx.$$

$= \frac{2}{\pi}$ [x^2 [illegible] – 2x

$$= \frac{2}{\pi}\left(x^2 \cdot \frac{\sin nx}{n}\right)_0^{\pi} - \left(2x \cdot x - \frac{\cos nx}{n^2}\right)_0^{\pi} + \left(2 \times - \frac{\sin nx}{n^3}\right)_0^{\pi}$$

$$= \frac{2}{\pi}\left[x^2 \cdot \sin\frac{nx}{n}\right]_0^{\pi} + \left[2x \times \frac{\cos nx}{n^2}\right]_0^{\pi} - \left[2 \cdot \frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 \cdot \frac{\sin(n\pi)}{n} - 0 \times \sin 0\right] + \left[2\pi \times \frac{\cos n\pi}{n^2} - 2 \times 0 \times \cos 0\right]$$

$\sin n\pi = 0$

$$- \left(2 \times \frac{\sin n\pi}{n^3} - 2 \times \sin 0\right)$$

$$= \frac{2}{\pi}\left[(0-0) + \left(2\pi \times \frac{(-1)^n}{n^2} - 0\right) - (0-0)\right]$$

$$= \frac{2}{\pi} \times 2\pi \times \frac{(-1)^n}{n^2} = \underline{\underline{\frac{4(-1)^n}{n^2}}}$$

$\therefore b_n = \underline{\underline{0}}$

Then the Fourier series $f(x)$ become,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right) \qquad f(x) = x^2.$$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$x^2 = \frac{1}{2}\left(\frac{2}{3} \cdot \pi^2\right) + \sum_{n=1}^{\infty} 4\frac{(-1)^n}{n^2} \cdot \cos(nx)$$

$(\cos 0 = 1.)$

$(-1)^1 = -1.$

$(-1)^2 = 1.$

$(-1)^3 = -1$

$\cos(nx) =$

$x = 0$

$$0 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$-\frac{\pi^2}{3} = 4\left[-\frac{1}{1^2} + \frac{1}{12} - \frac{1}{32} + \cdots\right]$$

$$-\frac{\pi^2}{12} = \left\{-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right.$$

$$\frac{1}{1^2} \pm \frac{1}{2^2} + \frac{1}{3^2} - \frac{\pi^2}{12} = 0 \Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} = \frac{\pi^2}{12} \text{ — ②}$$

23/2/16 $\underline{x = \pi}$

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{(-1)^n\,(-1)^n}{n^2}\right]$$ → even $(-1)^1 \times (-1$ $(-1)^2 \times ($

$$\pi^2 = \frac{\pi^2}{3} + \left\{4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right]\right.$$

$$\frac{\pi^2 - \pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

$$\frac{2\pi^2}{3} \times \frac{1}{4} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{2\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots\right]$$

~~π^2~~ $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\cdots\cdots = \frac{\pi^2}{6} \quad \text{——} ①$$

① + ② ⇒

$$\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2}\cdots\right] + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{6\pi^2 + 12\pi^2}{72}$$

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\overset{36}{18\pi^2}}{\underset{36}{72}}$$

4

$$2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

2) Find the Fourier series expansion of the function $f(x) = x - x^2$ from $x = -\pi$ to $x = \pi$ and hence show that $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{12}$

ans.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\, dx.$$

$$= \frac{1}{\pi}\left[\underset{\text{odd}}{\int_{-\pi}^{\pi} x\, dx} - \int_{-\pi}^{\pi} x^2\, dx\right]$$

$$= \frac{1}{\pi}\left[0 - 2\int_0^{\pi} \cancel{f(x)}\, x^2\, dx\right]$$

$$= \frac{1}{\pi}\left[-2\left(\frac{x^3}{3}\right)_0^{\pi}\right]$$

$$= \frac{1}{\pi} \times -2\ \frac{\pi^3}{3} = \underline{\underline{-\frac{2}{3}\pi^2}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2) \cos(nx)\, dx.$$

$$= \frac{1}{\pi}\left[\underset{\text{odd . even}}{\int_{-\pi}^{\pi} x \cos(nx)\, dx} - \int_{-\pi}^{\pi} x^2 \cos(nx)\, dx\right]$$

$\left[\int \text{even}\, dx = 2\int_0 f(x)\, dx\right]$

$\left[\int \text{odd} = 0\right]$

$$= \frac{1}{\pi}\left[0 - 2\int_0^{\pi} x^2 \cdot \cos(nx)\, dx.\right]$$

$\int u\ v\ dx = u v_1 - u' v_2 + u'' v_3 - \cdots$

$u = x^2$

$v = \cos nx$

$$= -\frac{2}{\pi}\left[\left(x^2 \cdot \frac{\sin(nx)}{n}\right)_0^{\pi} - \left(2x \cdot -\frac{\cos nx}{n^2}\right)_0^{\pi} + \left(2 \cdot -\frac{\sin(nx)}{n^3}\right)_0^{\pi}\right]$$

$$= \frac{-2}{\pi}\left[\left(\pi^2 \cdot \frac{\sin n\pi}{n} - 0\right) + \left(2\pi \frac{\cos n\pi}{n^2} - 0\right) - \left(2 \times \frac{\sin n\pi}{n^3}\right)\right]$$

$\sin n\pi = 0$, $\cos n\pi = (-1)^n$

$$= \frac{-2}{\pi} \times 2\pi \frac{(-1)^n}{n^2} = -4 \times \frac{(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \cancel{f(x)}\ \cancel{\sin}\ (x - x^2)\ \sin(nx)\, dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx) dx - \int_{-\pi}^{\pi} x^2 \sin(nx) dx.$$

odd . odd. even . odd.

$$= \frac{1}{\pi}\left[2\int_{0}^{\pi} x \sin(nx) dx - 0\right]$$

$$= \frac{2}{\pi} \left[\left(x \times -\frac{\cos(nx)}{n}\right)_0^{\pi} - \left(1 \times -\frac{\sin(nx)}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-\left(x \frac{\cos nx}{n}\right)_0^{\pi} + \left(\frac{\sin(nx)}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi} - \left(\pi \frac{\cos n\pi}{n} - 0\right) + \left(\frac{\sin n\pi}{n^2}\right)$$

$$= \frac{2}{\pi} - \pi \frac{(-1)^n}{n} = -2 \frac{(-1)^n}{n}$$

$= \frac{2(-1)^n \times (-1)}{n}$ [illegible]

Then, Required Fourier series

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

~~F(x) =~~

$$(x-x^2) = \frac{1}{2}\left(\frac{-2}{3}\pi^2\right) + \sum_{n=1}^{\infty} \frac{-4\,(-1)^n}{n^2}\cos(nx) + \sum_{n=1}^{\infty} \frac{-2\times(-1)^n}{n}\sin(nx)$$

$$(x-x^2) = \frac{-\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(nx) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx) \quad \text{(I)}$$

putting $x=0$ in equation (I) and expanding.

$$0 = \frac{-\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos(n\times0) - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin(n\times0)$$

$$0 = \frac{-\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$n = 1, 2, 3 \ldots$

$(-1)^1 = -1 \quad (-1)^3 = -1$

$(-1)^2 = 1 \quad\quad\quad\quad 3^2$

$$0 = \frac{-\pi^2}{3} - 4\left[\frac{(-1)}{1^2} + \frac{1}{2^2} - \frac{1}{3^2}\cdots\right]$$

$$\frac{\pi^2}{3} = -4\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right]$$

$$\frac{-\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} \Rightarrow -\frac{\pi^2}{12} + \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} = 0$$

$$\therefore \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}.$$

3) obtain the Fourier series of $f(x) = |x|$ in the interval $(-\pi, \pi)$. Hence show that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

ans: $F(-x) = |-x| = |x| = x$

$F(-x) = f(x) \Rightarrow f(x)$ is even.

Since $f(x)$ is even a_0, a_n exist, $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$= \frac{1}{\pi}\int^{\pi} |x|\,dx. \quad (|x| \text{ is even})$$

$|x| = x$ in $(0,$

$$= \frac{1}{\pi} 2\int_{0}^{\pi} x \, dx$$

$$= \frac{1}{\pi} 2 \left(\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \cancel{2} \frac{\pi^2}{\cancel{2}} = \underline{\pi}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos \cancel{\left(\frac{n\pi x}{\pi}\right)} dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| . \cos(nx) \, dx.$$

even. even = even

$$\cancel{= \frac{1}{\pi} \left(x \times \frac{\sin(nx)}{n}\right)_0 - \left(1 \times \frac{-\cos(nx)}{n^2}\right)}$$

$$= \frac{1}{\pi} 2 \int_0^{\pi} x \cos(nx) \, dx$$

$$= \frac{2}{\pi} \left(x \times \frac{\sin(nx)}{n}\right]_0^{\pi} - \left(1 \times -\frac{\cos(nx)}{n^2}\right)_0^{\pi}$$

$$= \frac{2}{\pi} \left[\left(x \frac{\sin(nx)}{n}\right)_0^{\pi} + \left(\frac{\cos(nx)}{n^2}\right)_0^{\pi} \right]$$

$$= \frac{2}{\pi} \left[\left(\pi \cdot \frac{\sin(n\pi)}{n} - 0\right) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos(n0)}{n^2}\right) \right]$$

$$= \frac{2}{\pi} \left(0 + \cancel{\cos n}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)\right)$$

$$= \frac{2}{\pi} \cdot \left[\frac{(-1)^n - 1}{n^2}\right]$$

~~$|x| = x$~~ $($

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$\therefore$ The Fourier series: ~~$f(x)$~~ =

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n - 1}{n^2}\right] \cos(nx) \quad (|x| = x, (0\ \pi))$$

$$f(x) = |x| = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx$$

$$x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx \quad (1)$$

when $x = 0$,

$$0 = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos 0$$

$$0 = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[-2 - 0 - \frac{2}{3^2} - 0 - \frac{2}{5^2}\right]$$

$$0 = \frac{\pi}{2} + \frac{2 \times -2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{2} = 4\left[1 + \frac{1}{3^2} + \frac{1}{5^2}\right] \Rightarrow \frac{\pi^2}{8} = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right]$$

$$\left[1 + \frac{1}{3^2} + \frac{1}{5^2} \cdots\right] = \frac{\pi^2}{8} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \underline{\underline{\frac{\pi^2}{8}}}$$

) Find the Fourier series of the function $f(x) = x$ in the range $-\pi < x < \pi$ and hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

: Ther given function $f(x) = x$ is an odd functi is the interval $(-\pi, \pi)$

$\therefore$ ~~$b_n = 0$~~ $a_0 = 0$, $a_n = 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int^{\pi} f(x)\sin x\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin(nx)\, dx.$$

pronecker formula

$\int u \cdot v \, dx =$

$u v_1 - u' v_2 + u'' v_3$

$$= \frac{2}{\pi}\left(x \times -\frac{\cos nx}{n}\right)_0^{\pi} - \left(1 \times -\frac{\sin nx}{n^2}\right)_0^{\pi}$$

$$= \frac{2}{\pi} - \left[\frac{x \cos(nx)}{n}\right]_0^{\pi} + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi} - \left(\pi \frac{\cos n\pi}{n} - 0\right) + \left[\frac{\sin n\pi}{n^2} - \frac{\sin 0}{n^2}\right]$$

$$= -\frac{2}{\pi} \times \frac{\pi (-1)^n}{n} = \underline{\underline{\frac{-2(-1)^n}{n}}} = \underline{\underline{\frac{2(-1)^{n+1}}{n}}}$$

$\therefore$ The Fourier series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$x = \sum_{n=1}^{\infty} \frac{-2 \times (-1)^{n}}{n} \sin(nx) \quad \text{———} \text{①}$$

$\sin 90 =$

$x =$

putting $x = \frac{\pi}{2}$ & expanding

$$\frac{\pi}{2} = \sum_{n=1}^{\infty} \frac{(2)(-1)^{n+1}}{n} \sin n\frac{\pi}{2}$$

$\frac{(-2)(-1)}{1} = 2$

$(-2)(-1)^2$

$$\frac{\pi}{2} = 2\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - -\right]$$

(No even ter

$n \neq 2, 4, 6$

$\sin nx \; ; \; x = \pi/$

$n=1, \sin \pi/2 = 1$

$n=2, \sin \pi = 0$

$$\frac{\pi}{4} = \left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - - -\right]$$

$$\underline{\left(1 - \frac{1}{3} + \frac{1}{5} - - -\right) = \frac{\pi}{4}}$$

[purely this is
a sine series]

* Fourier Cosine Series [-L, L]

If f(x) is an even function defined in the interval (-L, L), then $f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)$ is an even function and $f(x) \cdot \sin\left(\frac{n\pi x}{L}\right)$ is an odd function. Then by Oeulers formula,

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx.$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad \text{and}$$

$$b_n = 0$$

Then the fourier series f(x) is reduces to

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

2/2/16 Fourier Sine Series [-L, L]

*

Let f(x) be an odd function in the interval [-L, L] then $f(x) \cdot \cos\left(\frac{n\pi x}{L}\right)$ is an odd function and $f(x) \cdot \sin\left(\frac{n\pi x}{L}\right)$ is an even function.

Then by Oeuler's formula $a_0 = 0$, $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$$ and the Corresponding Fourier series reduces to

$$F(x) = \sum_{n=0}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

*Fourier cosine Series [-π, π]

Let f(x) be an even function in the interval [-π, π]

$$\boxed{\frac{n\pi x}{L} \rightarrow nx}$$

Then F(x); cos n(x) is an even function. and F(x)· sin(nx) is an odd function.

Cosine → even function

sine → odd

Then By Oeuler's formula,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} F(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

$$b_n = 0$$

Then the fourier series, f(x) reduces to

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

* Fourier Sine Series $[-\pi, \pi]$

Let $f(x)$ be an odd function in the interv $[-\pi, \pi]$ then $f(x).\cos(nx)$ is an odd functi and $f(x)\sin(nx)$ is an even function. Then by Oeuler's Formula $a_0 = 0$, $a_n = 0$ and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x).\sin(nx)\,dx$. Then the Fourier series of $f(x)$ reduces to the form

$$F(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

5) obtain the fourier series $f(x) = |\sin x|$ in the interval $-\pi \le x \le \pi$

$f(-x) = f(x) \Rightarrow$

$f(-x) = -f(x) =$

ans: $f(x) = |\sin x|$

$$f(-x) = |\sin(-x)| = |-\sin x| = |\sin x| = f(x)$$

$\sin(-x) = $ Si

$\cos -x = $ c

function is even.

since $f(x)$ is an even function $\boxed{|\sin x| = \sin x, (0}$

~~f(x)~~ i $b_n = 0$.

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} |\sin x| \; dx.$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \; dx.$$

~~cos~~ $-(-1 -1)$.

$+\frac{2 \times 2}{\pi}$

$$= \frac{2}{\pi} \left(-\cos x\right)_0^{\pi}$$

$$= \frac{2}{\pi} \left((-\cos\pi + \cos\pi\right)\left(- \; -\cos 0 + \cos 0\right)$$

$$= \frac{2}{\pi} \cancel{(-1+1)} = [0 - (1+1))]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \;\; \cos(nx)\,dx.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} |\sin x| \;\; \cos(nx)\,dx.$$

$$= \frac{1}{\pi} \int_0^{\pi} 2\sin x \cdot \cos(nx) \; dx.$$

$(\cancel{A=1}\; A=1 \;\; B=n)$

$$= \frac{\cancel{2}}{\pi} \;\; \cancel{\sin(x+nx) + \sin(x-nx)}$$

~~f(u·v)dx~~ =

$$\boxed{\begin{gathered} 2\sin x \cos y = \\ \sin(\underset{A+B}{x+y}) + \sin(\underset{A-B}{x-y}). \end{gathered}}$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin(1+n)x + \sin(1-n)x\,dx.$$

$A = x$, $B = nx$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x + \sin(nx) + \sin x - \sin(nx) \quad dx.$$

$= \frac{1}{\pi}\left(2 - \frac{(-1)^n}{n} + 1 + \cancel{2} + \frac{(-1)^n}{n} - \frac{1}{n}\right)$

$$= \frac{1}{\pi}\left[\left(\frac{-\cos(1+n)x}{1+n}\right)_0^{\pi} + \left(-\frac{\cos(1-n)x}{(1-n)}\right)_0^{\pi}\right]$$

$$= -\frac{1}{\pi}\left(\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1+n)0}{1+n}\right) + \left[\frac{\cos(1-n)\pi}{1-n} - \frac{\cos(1-n)0}{1-n}\right]$$

$$= \frac{-1}{\pi}\left[\left(\frac{(-1)^{1+n}}{1+n} - \frac{1}{1+n}\right) + \left(\frac{(-1)^{1-n}}{1-n} - \frac{1}{1-n}\right)\right]$$

$$= \frac{-1}{\pi}\left[\left(\frac{(-1)^{1+n}}{1+n} + \frac{(-1)^{1-n}}{1-n}\right) - \left(\frac{1}{1+n} + \frac{1}{1-n}\right)\right]$$

$(-1)^1$ 1^{1+n} next −

$\frac{2[(-1)^n+1]}{\pi(1-n)^2}$

$$= \frac{1}{\pi}\left[\frac{(-1)^{2+n}}{1+n} + \frac{(-1)^{2-n}}{1-n}\right] + \left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$(-1)^{2+n} \Rightarrow$

$(-1)^2 = 1$

$(-1)^n$

$= (-1)^n$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n}\right] + \left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

* Imp

$(-1)^{2+n} = (-1)^n$

$n=1$ $(-1)^3 = -1$

$n=2$ $1 = 1$

$n=3$ $-1 = -1$

$n=4$ $1 = 1$

$$= \frac{1}{\pi}\left[\frac{1[(-1)^n + 1]}{1+n} + \frac{1[(-1)^n + 1]}{1-n}\right]$$

$$= \frac{(-1)^n + 1}{\pi}\left[\frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{(-1)^n + 1}{\pi} \cdot \frac{1-\cancel{n} + 1 + \cancel{n}}{(1+n)(1-n)} = \frac{(-1)^n + 1}{\pi}\,\frac{2}{1-n^2} \quad \boxed{n \neq 1}$$

Now, we are going to find, a_1 separately i.e,

$$a_1 = \frac{2}{\pi}\int_0^\pi f(x)\cos(1x)\,dx. \qquad n=1$$

$|\sin x| = \sin x\ (0,\pi)$

$\int u\,v.\,dx =$

$$= \frac{2}{\pi}\int_0^\pi \sin x \quad \cos(1x)\ dx$$

$$= \frac{1}{\pi}\left(\sin x\ \sin nx\right)_0^\pi \quad \cos xn$$

$$= \frac{1}{\pi}\int_0^\pi 2\sin x \quad \cos x \quad dx$$

$$= \frac{1}{\pi}\int_0^\pi \sin 2x \quad dx.$$

$$= \frac{1}{\pi}\left[\frac{-\cos 2x}{2}\right]_0^\pi$$

$$= \frac{1}{\pi} - \left[\frac{\cos 2\pi}{2} - \frac{\cos 0}{2}\right] \qquad n1$$

$$= \frac{-1}{2\pi} \quad \underline{\underline{\left[(-1)^n - 1\right]}} \qquad \begin{matrix} n=2 \\ (-1)^2. \end{matrix}$$

$$= \frac{-1}{2\pi}(1-1) \quad \underline{\underline{=0}}$$

Fourier series expansion for the given function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)\,dx$$

$$|\sin x| = \frac{4}{2\pi} + \sum_{n=2}^{\infty} \frac{2\left((-1)^n + 1\right)}{\pi(1-n^2)}\cos(nx)$$

$\left(\frac{2}{\pi}\right)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$|\cos x| = \frac{4}{2\pi} + \sum_{n=1}^{\infty} \frac{4\cos\left(\frac{n\pi}{2}\right)}{\pi(1-n^2} \cos(nx)$$

$$\cos x = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{4\cos\left(\frac{n\pi}{2}\right)}{\pi(1-n^2)} \cos(nx)$$

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Fourier Series expression in $(0, 2L)$

* $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx.$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cdot\cos\left(\frac{n\pi x}{L}\right)dx.$$

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx.$$

Note:

If the given range is $(-L, L)$ the formula is same and the limits of integration will change to $(-L, L)$

even - Fourier Cosine series

Odd - Fourier Sine series

- when $f(x)$ is even, (Fourier Cosine series)

then $b_n = 0$.

The Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\therefore a_0 = \frac{2}{L}\int_0^{L} f(x)\,dx.$$

$$a_n = \frac{2}{L} \int_0^{L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

- When $f(x)$ is odd i.e. fourier sine series.

where $a_0 = 0$,

$a_n = 0$.

$$\therefore b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

then the fourier series

$$f(x) = \frac{2}{L} \int_0 \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) dx.$$

* Fourier Series expression in $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

where $a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x) dx.$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos(nx) dx.$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \sin(nx) dx.$$

Note: If $f(x)$ is given in the interval $(-\pi, \pi)$ then the formula's are same as above and the limits of integration are changed of $-\pi, \pi)$

$\frac{n\pi x}{l} \to nx.$

- when $f(x)$ is equal to an even function.

then $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx.$$

Then the fourier series,

~~$f(x) = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$~~

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

- when $f(x)$ is equal to an odd function

$$a_0 = 0, \quad a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cdot\sin(nx)\,dx.$$

$$\therefore F(x) = \sum_{n=1}^{\infty} b_n \sin(nx)\,dx$$

Q1) Obtain the Fourier series expansion for th function $f(x) = \frac{\pi - x}{2}$, $0 < x < 2$.

$0 < x < 2l$

$\Downarrow$ $2l = 2$

$l = 1$

ans.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

[a_0, a_n, b_n → eulers constants]

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$$

$$= \frac{1}{1}\int_0^{2L} \frac{\pi - x}{2}\,dx.$$

$$= \frac{1}{2}\int_0^{2\cancel{L}\times 1} \pi - x \quad dx.$$

$$= \frac{1}{2}\left(\pi(x)_0^{2\times 1} - \left(\frac{x^2}{2}\right)_0^{2\cancel{L}}\right)$$

$$= \frac{1}{2}\left(\pi \times 2\cancel{L} - \frac{\cancel{4L^2}}{2}\right)$$

$$= \cancel{\frac{1}{2}}\ \cancel{(2\pi L - 2L^2)}\ \frac{1}{2}(2\pi - 2) = \underline{\underline{\pi - 1}}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx. \qquad (L=1)$$

$$= \frac{1}{1}\int_0^{2}\left(\frac{\pi - x}{2}\right)\cdot\cos(n\pi x)\ dx$$

$$= \frac{1}{2}\int_0^{2}\underset{u}{(\pi - x)}\cdot\underset{v}{\cos}(n\pi x)\ dx.$$

$$\boxed{\begin{array}{l}\text{Applying prohecker's for}\\ \int uv\,dx = \\ uv_1 - u'v_2 + u''v_3\end{array}}$$

$$\boxed{\begin{array}{l}u = \pi - x\\ v = \cos(n\pi x)\end{array}}$$

$$= \frac{1}{2}\int\left[\left((\pi - x)\ \frac{\sin(n\pi x)}{n\pi}\right)_0^2 + \left(1\times -\frac{\cos(n\pi x)}{(n\pi)^2}\right)_0^2\right]$$

$$= \frac{1}{2}\left[(\pi - x)\left(\frac{\sin n\pi x}{n\pi}\right)\right]_0^2 - \left(\frac{\cos(n\pi x)}{n^2\pi^2}\right)_0^2\Bigg]$$

$$= \frac{1}{2}\left[\left((\pi-2)\ \frac{\sin(2n\pi)}{n\pi}\right) - \left((\pi - 0)\ \frac{\sin(n\pi\times 0)}{n\pi}\right) - \left(\frac{\cos 2n\pi}{n^2\pi^2} - \frac{\cos n\pi\times 0}{n^2\pi^2}\right)\right]$$

$$\cos 2n\pi \quad (-1)^{2n} = 1$$

$$= \frac{1}{2}\left[((\pi - 2))\times 0 - \pi \times 0\right) - \left(\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right)$$

$$\boxed{\begin{array}{l}\cos n\pi = (-1)^n\\ \cos 2n\pi = (-1)^{2n} =\end{array}}$$

$$= \frac{1}{2} \cdot \left[-\left(\frac{1}{n^2\pi^2} - \frac{1}{n^2\pi^2} \right) \right]$$

$$\therefore a_n = \underline{\underline{0}}$$

$$b_n = \frac{1}{L} \int_0^{2L} f(x) \cdot \frac{\sin(n\pi x)}{L} \, dx.$$

$$= \frac{1}{2} \int_0^2 \underset{u}{(\pi - x)} \cdot \underset{v}{\sin(n\pi x)} \, \underset{dx}{dx}.$$

$$= \frac{1}{2}\left[-\left((\pi - x) \cdot \frac{\cos n\pi x}{n\pi} \right)_0^2 - \left(-1 \cdot x \cdot \frac{\sin n\pi x}{n^2\pi^2} \right)_0^2 \right]$$

$$= \frac{1}{2}\left[-\left[(\pi - x) \frac{\cos(n\pi x)}{n\pi} \right]_0^2 + \left(\frac{\sin(n\pi x)}{n^2\pi^2} \right)_0^2 \right]$$

$$= \frac{1}{2} - \left[(\pi - 2) \frac{\cos(2n\pi)}{n\pi} - (\pi - 0)\left(\frac{\cos n\pi \times 0}{n\pi} \right) \right] + \left[\frac{\sin 2n\pi}{n^2\pi^2} - \frac{\sin n\pi x}{n^2\pi^2} \right.$$

$$= -\frac{1}{2}\left[\left((\pi - 2) \frac{1}{n\pi} \right) - \left(\pi \cdot \frac{1}{n\pi} \right) + (0 - 0) \right]$$

$$= -\frac{1}{2}\left[\frac{\pi - 2}{n\pi} - \frac{1}{n} \right]$$

$$= -\frac{1}{2} \times \frac{n\pi - 2n - n\pi}{n^2 \pi}$$

$$= \underline{\underline{\frac{1}{n\pi}}}$$

∴ The fourier series expression for.

$$f(x) = \frac{\pi - 1}{2} + \sum_{n=1}^{\infty} 0 \cdot \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

$L = 1$

$$\frac{\pi - x}{2} = \frac{\pi - 1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin(n\pi x)$$

2) Obtain the fourier series for

$$f(x) = \cancel{1} \begin{cases} 1, & \overset{0 < x < L}{0 < x < 1} \\ 2, & \underset{(L < x < 2L)}{1 < x < 2} \end{cases}$$

$2L = 2$

$\therefore \underline{\underline{L = 1}}$

ans

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\therefore a_0 = \frac{1}{L}\int_0^{2L} f(x)\, dx.$$

$$= \frac{1}{1}\left[\int_0^1 1\, dx + \int_1^2 2\, dx\right]$$

$$= \left[(x)_0^1 + (2x)_1^2\right]$$

$$= 1 + (2 \times 2 - 2)$$

$$= 1 + (4 - 2) = 1 + 2$$

$$= \underline{\underline{3}}$$

$$\therefore a_n = \frac{1}{L}\int_0^{2L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx.$$

$$= \frac{1}{1}\left[\int_0^1 1 \cos(n\pi x)\, dx + \int_1^2 2 \cos(n\pi x)\, dx\right.$$

$$= \left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 + 2\left[\frac{\sin n\pi x}{n\pi}\right]_1^2$$

$$= \left(\frac{\sin(n\pi)}{n\pi} - \frac{\sin 0}{n\pi}\right) + 2\left(\frac{\sin 2n\pi}{n\pi} - \frac{\sin n\pi}{n\pi}\right)$$

$$= \underline{\underline{0}}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\,\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\left[\int_0^1 1\,\sin(n\pi x)\,dx + \int_1^2 2\,\sin(n\pi x)\,dx\right]$$

$$-\left(\frac{\cos n\pi x}{n\pi}\right)_0^1 + -2\left[\frac{\cos n\pi x}{n\pi}\right]_1^2$$

$$= -\left(\frac{\cos n\pi}{n\pi} - \frac{\cos 0}{n\pi}\right) - 2\left[\frac{\cos 2n\pi}{n\pi} - \frac{\cos n\pi}{n\pi}\right]$$

$$= -\left(\frac{(-1)^n}{n\pi} - \frac{1}{n\pi}\right) - 2\left[\frac{1}{n\pi} - \frac{(-1)^n}{n\pi}\right]$$

$$= a\left(\frac{(-1)^n + 1}{n\pi}\right) - \frac{2}{n\pi}$$

$$= \frac{-(-1)^n}{n\pi} + \frac{1}{n\pi} - \frac{2}{n\pi} + \frac{2(-1)^n}{n\pi}$$

$$= \left[\frac{-(-1)^n + 1}{n\pi} - \frac{2 + 2(-1)^n}{n\pi}\right]$$

$$= \left[\frac{2(-1)^n - (-1)^n - 2 + 1}{n\pi}\right] = \underline{\underline{\frac{(-1)^n - 1}{n\pi}}}$$

$\therefore$ The fourier series.

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} 0 \times \cos(n\pi x) + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n\pi}\right)\sin\left(\frac{n\pi x}{1}\right)$$

$$\underline{\underline{f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n\pi}\right)\sin(n\pi x)}}$$

3) If $f(x) = \left(\frac{\pi - x}{2}\right)^2$, 0 to 2π, show that

HW. $f(x) = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$?. $\Rightarrow$

4) obtain the fourier series of $f(x) = e^{-x}$, $0 \leq x \leq$

$0, < x < 2$

ans,

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x}\, dx.$$

$$= \frac{1}{\pi} \left(-e^{-x}\right)_0^{2\pi}$$

$$= \frac{-1}{\pi} \left(e^{-2\pi} - e^{-0}\right).$$

$$= \frac{-1}{\pi} \left(e^{-2\pi} - 1\right).$$

$$= \frac{(1 - e^{-2\pi})}{\pi} \quad a \quad \frac{1}{}$$

imp

$\int e^{ax} \cos bxd$

$= \frac{e^{ax}}{a^2+b^2} [a \cos b$

$b \sin$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos(nx)\, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^{-x} \cdot \cos(nx)\, dx.$$

$$= \frac{1}{\pi} \left[\frac{e^{-x}}{(-1)^2 + n^2} \cdot (-\cos(nx) + n \sin(nx)) \right]_0^{2\pi}$$

Here, $a = -1$
$b = n$

$$= \cancel{\frac{-1}{\pi} \left[\frac{e^{-x}}{1^2 + n^2} \cdot (\cos nx + n \sin(nx)) \right]_0^{2\pi}}$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-2\pi}}{1^2+n^2}(-\cos(2n\pi)+n\sin(2n\pi))\right)-\left(\frac{e^{-0}}{1^2+n^2}(\cos 0+n\sin 0)\right)\right]$$

$$\left[\cos 2n\pi = (-1)^{n}(-1)^{2}_{\;1}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-2\pi}}{1^2+n^2}\cdot -1\right) - \left(\frac{1}{1^2+n^2}\cdot -1\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{-e^{-2\pi}}{1^2+n^2}+\frac{1}{1^2+n^2}\right) = \frac{1}{\pi}\cdot\frac{1}{1^2+n^2}\left(1-e^{-2\pi}\right)$$

$$= \frac{1\cdot\left(1-e^{-2\pi}\right)}{\pi\left(1^2+n^2\right)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\ \sin(nx)\,dx.$$

$$= \frac{1}{\pi}\int_0^{2\pi} e^{-x}\ \sin(nx)\,dx.$$

$$\left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx-b\cos bx\right]\right]$$

$$a=-1,\ b=n$$

$$= \frac{1}{\pi}\left[\frac{e^{-x}}{1^2+n^2}\left(-\sin nx-n\cos nx\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-2\pi}}{1^2+n^2}\left(-\sin 2n\pi-n\cos 2n\pi\right)\right)-\left(\frac{e^{-0}}{1^2+n^2}(-\sin 0-n\cos 0)\right)\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{-2\pi}}{1^2+n^2}\times -n\times 1\right)-\left(\frac{1}{1^2+n^2}\ -n\times 1\right)\right]$$

$$= \frac{1}{\pi}\left(\frac{-e^{-2\pi}}{1^2+n^2}\times n+\frac{n}{1^2+n^2}\right)$$

$$= \frac{1}{\pi}\times\frac{n}{1^2+n^2}\left(1-e^{-2\pi}\right)$$

$$= \frac{n}{\pi\left(1^2+n^2\right)}\left(1-e^{-2\pi}\right)$$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$F(x) = \frac{(1-e^{-2\pi})}{2\pi} + \sum_{n=1}^{\infty} \frac{1-e^{-2\pi}}{\pi(1^2+n^2} \cos(nx) + \sum_{n=1}^{\infty} \frac{n(1-e^{-2\pi})}{1^2+n^2} \sin$$

5) obtain the fourier series of $f(x) = x \sin x$ $0 \le x \le 2\pi$ and hence reduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} \ldots = \frac{\pi - 2}{4}$$

ans:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} f(x)\, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} \overset{u}{x}\, \overset{v}{\sin x} \cdot \overset{}{dx}.$$

$$= \frac{1}{\pi}\left(\left(x \cdot -\cos x\right)_0^{2\pi} - \left(1 \times \sin x\right)_0^{2\pi}\right)$$

$$= \frac{1}{\pi}\left[-(2\pi \cos 2\pi - 0) - (\sin 2\pi - \sin 0)\right]$$

$$= \frac{-2\pi \times 1}{\pi} = \underline{\underline{-2}}$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos(nx)\, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} x\left[\sin x \cdot \cos(nx)\, dx\right]$$

$$= \frac{1}{\pi} \cdot \frac{x}{2} \int_0^{2\pi} \left[\sin(1+n)x + \sin(1-n)x\right] dx$$

$2\sin A \cos B =$
$\sin(A+B) + \sin$

$\sin A \cos B = \frac{1}{2}$
$\frac{1}{2} \sin(A+B) + \sin($

$n - x = n$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} \underset{u}{x} \cdot \underset{v}{\sin(1+n)x}\, dx + \int_0^{2\pi} \underset{u}{x} \cdot \underset{v}{\sin(1-n)x}\, dx\right]$$

$$= \frac{1}{2\pi}\left[\left(x \cdot \frac{-\cos(1+n)x}{1+n}\right)_0^{2\pi} - \left(1 \times -\frac{\sin(1+n)x}{(1+n)^2}\right)_0^{2\pi}\right] +$$

$$\left[\left(x \times \frac{-\cos(1-n)x}{1-n}\right)_0^{2\pi} - \left(1 \times -\frac{\sin(1-n)x}{(1-n)^2}\right)_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[-\left(2\pi\, \frac{\cos(1+n)2\pi}{1+n} - 0\right) + \left(\frac{\overset{0}{\sin(1+n)2\pi}}{(1+n)^2} - \overset{0}{\sin 0}\right)\right] +$$

$$\left[-\left(2\pi \times \frac{\cos(1-n)2\pi}{(1-n)} - 0\right) + \left(\frac{\sin(1-n)2\pi}{\underset{0}{(1-n)^2}} - \overset{}{\underset{0}{\sin 0}}\right)\right]$$

$$\frac{1}{\cancel{2\pi}} \quad - 2\cancel{\pi} \times \frac{(\cos 2\pi + 1)}{\cancel{n}} - 2\pi\, \frac{\cos 2\pi - 1}{\cancel{n}}.$$

$$\boxed{\begin{aligned} &\cos n\pi = (-1)^n \\ &\cos(1+n)2\pi = \\ &(-1)^{(1+n)2} \\ &= (-1)^{2+2n} \\ &= 1 \text{ +ve} \end{aligned}}$$

$$= \frac{1}{2\pi}\left(-2\pi \times \frac{1}{1+n} - 2\pi\, \frac{1}{1-n}\right]$$

$$= \frac{-2\pi}{2\pi}\left(\frac{1}{1+n} + \frac{1}{1-n}\right)$$

$$\boxed{\begin{aligned} &\cos(1-n)2\pi = \\ &(-1)^{(1-n)2} \\ &= (-1)^{2-2n} \\ &= (-1)^0 = 1 \\ &\text{+ve.} \end{aligned}}$$

$$= -1 \times \frac{(1-\cancel{n}) + (1+\cancel{n})}{(1+n)(1-n)}$$

$$= -\frac{2}{1-n^2}, \quad n \neq 1.$$

Now we $(n=1)$ are finding a_1 separately as.

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx.$$

$$\boxed{2\sin x \cos x = \sin 2x}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x \cos x)\, dx.$$

$$= \frac{1}{}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \underset{u}{x} \cdot \underset{v}{\sin 2x}\, dx .$$

$$= \frac{1}{2\pi} \left[\left(x \cdot \frac{-\cos 2x}{2} \right)_0^{2\pi} - \left(1 \times \frac{-\sin 2x}{4} \right)_0^{2\pi} \right]$$

$$= \frac{1}{2\pi} \left[-\left(2\pi \frac{\cos 2\pi}{2} - 0 \right) + \left(\frac{\sin 2 \times 2\pi}{4} - \sin 0 \right) \right]$$

$$= \frac{1}{2\cancel{\pi}} \times -\cancel{\pi} \times 1 . = \underline{\underline{\frac{-1}{2}}}$$

$2\sin A \sin B$ =
$\cos(A-B)-co$

$\sin A \sin B =$
$\frac{1}{2}(\cos(A-B)-($
$A = 1, B=$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx .$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \ \sin(nx) \ dx .$$

$$= \frac{1}{2\pi} \int_0^{2\pi} x , \left[\cos(1-n)x - \cos(1+n)x \right] dx .$$

$$= \frac{1}{2\pi} \left[\int_0^{2\pi} \underset{u}{x} \cdot \underset{v}{\cos(1-n)x}\, dx - \int_0^{2\pi} \underset{u}{x} \cdot \underset{v}{\cos(1+n)x}\, dx \right.$$

$$= \frac{1}{2\pi} \left[\left(x \cdot \frac{\sin(1-n)x}{1-n} \right)_0^{2\pi} - \left(1 \cdot \frac{-\cos(1-n)x}{(1-n)^2} \right)_0^{2\pi} \right] -$$

$$\left[\left(x \cdot \frac{\sin(1+n)x}{1+n} \right)_0^{2\pi} - \left(1 \cdot \frac{-\cos(1+n)x}{(1-n)^2} \right)_0^{2\pi} \right]$$

$$= \frac{1}{2\pi} \left[\left(2\pi \cdot \frac{\sin(1-n)2\pi}{1-n} - 0 \right) + \left(\frac{\cos(1-n)2\pi}{(1-n)^2} - 0 \right) \right] -$$

$$\left[\left(2\pi \cdot \frac{\sin(1+n)2\pi}{1+n} - 0 \right) + \left(\frac{\cos(1+n)2\pi}{(1-n)^2} - 0 \right) \right]$$

$$= \frac{1}{2\pi} \cdot \left(\frac{1}{(1-n)^2} - \frac{1}{(1-n)^2} \right)$$

$$= \frac{1}{2\pi} \times 0 \frac{2}{(1-n)^2} = \frac{1}{\pi (1-n)^2} \quad , n \neq 0$$

$$= \underline{\underline{0}} , \quad n \neq 1$$

So, we are finding b_1 separately as,
$(n=1)$

$$\boxed{\cos 2x = 1 - \sin^2 x} \Downarrow \boxed{\sin^2 x = \frac{1 - \cos 2x}{2}}$$

$$b_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \sin x \, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin^2 x \, dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \left(\frac{1 - \cos 2x}{2} \right) dx.$$

$$= \frac{1}{2\pi} \left[\int_0^{2\pi} x \, dx - \int_0^{2\pi} \underset{u \cdot v}{x \cos 2x} \, dx. \right]$$

$$= \frac{1}{2\pi} \left[\left(\frac{x^2}{2} \right)_0^{2\pi} - \left(x \cdot \frac{\sin 2x}{2} \right)_0^{2\pi} - \left(1 \times -\frac{\cos 2x}{4} \right)_0^{2\pi} \right]$$

$$= \frac{1}{2\pi} \left[\left(\frac{4\pi^2}{2} - 0 \right) - \left(2\pi \times \frac{\sin 2 \times 2\pi}{2} - 0 \right) + \left(\frac{\cos 2 \times 2\pi}{4} - \frac{\cos 2 \times 0}{4} \right) \right]$$

$$= \frac{1}{2\pi} \times \left[\frac{4\pi^2}{2} - 2\pi \times 0 + \frac{1}{4} - \frac{1}{4} \right]$$

$$= \frac{1}{2\pi} \times \left(\frac{4\pi^2}{2} - \frac{3}{4} \; 0 \right)$$

$$\frac{1}{4} - 1$$

$$\frac{1-4}{4} = \frac{-3}{4}$$

$$= \frac{4\pi^2}{4\pi} = \pi$$

$$= \pi$$

5/3/2016
Saturday

Half Range Series

* Half range cosine series

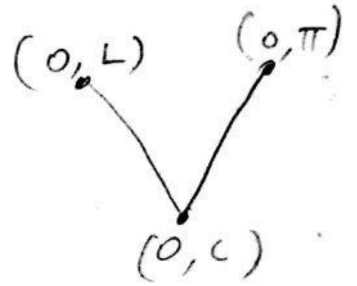
By solving partial differential equation, we come across with situation where we will have to express the initial condition in terms of sine and cosine series in a definite interval of (0, c), where 'c' can be replaced either by L or π.

If we want a fourier cosine series with the intial condition defined as an even function in the interval (-c, c). If we extend the the function f(x) by reflecting about the y axis (even function), i.e, $f(-x) = f(x)$, then the extended function is an even function and $b_n = 0$.

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx.$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\left(\frac{n\pi x}{c}\right)dx.$$

$$b_n = 0.$$



Then the Fourier series f(x) is corresponding to the half range cosine series become

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right)$$

* Half range sine series

If we want a fourier sine series in the interval (0, c) we extend the function as an odd function in the interval (-c, c). If we extend the function by reflecting about the origin. So, that $f(x) = -f(x)$; then the extended function is odd for which,

$$\boxed{\begin{aligned} a_0 &= 0 \\ a_n &= 0 \\ b_n &= \frac{2}{c}\int_0^c f(x)\ \sin\left(\frac{n\pi x}{c}\right)dx. \end{aligned}}$$

Then the Fourier Series $F(x)$ correspondin to the half range sine series become ~~F(x)~~.

$$\boxed{F(x) = \sum_{n=1}^{\infty} b_n\ \sin\left(\frac{n\pi x}{c}\right)}$$

1) Given the function, $F(x) = x$, $0 < x < \pi$

I ⓐ Find the half range Fourier Cosine series of $f(x)$.

ⓑ Find the half range fourier sine series of f

OR

II ⓐ Find the fouriera cosineseries of $f(x)$

ⓑ Find the fourier sine series of $f(x)$.

I ⓐ and II ⓐ

ans: Here we required to find a half range cosine series. ie, an even function in th interval $(0, \pi)$ then, $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\ dx.$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx.$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{2}{\pi} \times \frac{\pi^2}{2}$$

$$= \underline{\underline{\pi}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \; \cos\left(\frac{n\pi x}{\pi}\right) dx..$$

$$= \frac{2}{\pi}\int_0^{\pi} \underset{u.}{x} . \; \underset{v}{\cos(nx)} \; \underset{dx.}{dx.}$$

$$= \frac{2}{\pi}\left[\left(x \cdot \frac{\sin(nx)}{n}\right)_0^{\pi} - \left(1 \times \frac{-\cos(nx)}{n^2}\right)_0^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left(\pi \cdot \frac{\sin n\pi}{n} - 0\right) + \left(\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left(0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right)$$

$$(-1)^1 - 1 = -2$$

$$-2 \times 2 = \underline{\underline{\frac{-4}{\pi n^2}}}$$

$$= \underline{\underline{\frac{2}{\pi} \times \frac{(-1)^n - 1}{n^2}}} \quad \begin{cases} 0, & \text{even} \\ \frac{-4}{\pi n^2}, & \text{odd} \end{cases}$$

Then the half range fourier series,

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\pi}\right)$$

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cancel{a_n}\, a_n \cos(nx),$$

$$x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\times((-1)^n - 1)}{\pi n^2} \cos(nx)$$

$$x = \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{n^2} \cdot \cos(nx)$$

$\frac{(-1)^1-1}{1^2} =$

$\frac{(-1)^2-1}{2^2} =$

$\frac{(-1)^3}{3}$

$-1-1$

when $n = 1, 2, 3,$

$$x = \frac{\pi}{2} + \frac{2}{\pi}\left[(-2)\cos x + 0 \times \cos 2x + \frac{(-2)}{3^2}\cos 5x\right]$$

$$x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} \dots\right]$$

I (b) and II (b)

ans: Here the required series is a half rang series ie, the required ~~series~~ function is an odd function for which.

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{\pi}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi f(x)\sin\left(\frac{n\pi x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\int_0^\pi \underset{u}{x} \cdot \underset{v}{\sin(nx)}\,dx.$$

$$= \frac{2}{\pi}\left[\left(x \times \frac{-\cos(nx)}{n}\right)_0^\pi - \left(1 \times \frac{-\sin(nx)}{n^2}\right)_0^\pi\right]$$

$$= \frac{2}{\pi}\left[-\left(\frac{x\cos nx}{n}\right)_0^\pi + \left(\frac{\sin(nx)}{n^2}\right)_0^\pi\right]$$

$$= \frac{2}{\pi} - \left[\left(\pi\frac{\cos n\pi}{n} - 0\right) + \left(\frac{\sin n\pi}{n^2} - \sin 0\right)\right]$$

$$= \frac{2}{\pi}\left[\left(-\pi \times \frac{(-1)^n}{n}\right) + (0-0)\right]$$

$$(-2)(-1)^{2+1} = \frac{-2}{n}$$

$$= \frac{-2\cancel{\pi}}{\cancel{\pi}} \times \frac{(-1)^n}{n}.$$

$$= \underline{\underline{\frac{(-2)(-1)^n}{n}}} \quad = \quad \underline{\underline{\frac{2(-1)^{n+1}}{n}}} \; ; \; \begin{cases} \frac{-2}{n}, & \text{even} \\ \frac{2}{n}, & \text{odd} \end{cases}$$

Then the half range fourier series,

$$F(x) = \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n} \sin(nx)$$

$$\boxed{F(x = \sum_{n=1}^{\infty} bn \sin\left(\frac{n\pi x}{c}\right) \quad (c=\pi)}$$

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

$$2 \times (-1)^{1+1}$$

$$\frac{2 \times (-1)^{2+1}}{2} = \frac{-2}{2}$$

$$\frac{2 \times (-1)^{3+1}}{3} = \frac{2}{3}$$

$$x = \frac{2}{1} \sin x + \left(\frac{-2}{2}\right) \cdot \sin 2x + \left(\frac{2}{3}\right) \sin 3x \ldots\ldots$$

$$x = \underline{\underline{2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3}\right\}\ldots\ldots}}]$$

2) Find the Fourier cosine series as well as fourier sine series for $f(x) = x^2,\ 0 \le x \le c$ also sketch F(x) and two periodic function.

ans:

Half range cosine series
(even function)

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx.$$

$$a_0 = \frac{2}{c}\int_0^c x^2\,dx.$$

$$a_0 = \frac{2}{c}\left(\frac{x^3}{3}\right)_0^c$$

$$= \frac{2}{c}\times\frac{c^3}{3} \quad = \underline{\underline{\frac{2c^2}{3}}}.$$

$$a_n = \frac{2}{c}\int_0^c f(x)\ \cos\left(\frac{n\pi x}{c}\right) dx.$$

$$= \frac{2}{c}\int_0^c \overset{u}{x^2} .\ \overset{v}{\cos}\left(\frac{n\pi x}{c}\right) dx.$$

$$= \frac{2}{c}\left[\left(x^2 .\ \frac{\sin\left(\frac{n\pi x}{c}\right)}{\frac{n\pi}{c}}\right)_0^c - \left(2x \times \frac{-\cos\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)^2}\right)_0^c + \left(2\times\frac{-\sin\frac{n\pi x}{c}}{\left(\frac{n\pi}{c}\right)^3}\right)_0^c\right]$$

$$= \frac{2}{c}\left[\left(x^2 .\ \frac{\sin\frac{n\pi x}{c}}{\frac{n\pi}{c}}\right)_0^c + \left(2x\ \frac{\cos\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)^2}\right)_0^c - \left(2\ \frac{\sin\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)^3}\right.\right.$$

$$= \frac{2}{c}\left[\left(c^2 .\ \frac{\overset{\sin\frac{n\pi c}{c}}{\cancel{\sin \pi x c}}\big/\cancel{x}}{\frac{n\pi}{c}} - 0\right) + \left(2c\ \frac{\cos\left(\frac{n\pi c}{c}\right)}{\left(\frac{n\pi}{c}\right)^2} - 2\times 0\right) - \left(2\ \frac{\sin\left(\frac{n\pi c}{c}\right)}{\left(\frac{n\pi}{c}\right)^3} - 0\right)\right]$$

$$= \frac{2}{c}\left[\left(c^2 .\ \frac{\sin(n\pi)}{n\pi/c}\right) + \left(2c .\ \frac{\cos(n\pi)}{(n\pi/c)^2}\right) - \left(2\ \frac{\sin(n\pi)}{(n\pi/c)^3}\right.\right.$$

$$= \frac{2}{c}\left[\left(c^2 \cdot c\frac{\sin(n\pi)}{(n\pi)}\right) + \left(2c \times c^2 \times \frac{\cos(n\pi)}{(n\pi)^2}\right) - \left(2c^3\frac{\sin(n\pi)}{(n\pi)^3}\right)\right]$$

$$= \frac{2}{c}\left(0 + 2c^3\frac{\cos(n\pi)}{(n\pi)^2} - 0\right)$$

$$= \frac{4c^3}{c}\,\frac{\cos(n\pi)}{(n\pi)^2}$$

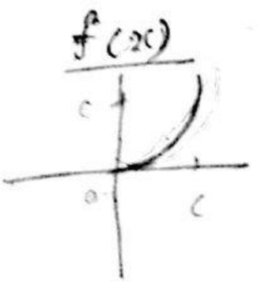
$$= \underline{\underline{\frac{4c^2 \times (-1)^n}{(n\pi)^2}}} = \underline{\underline{\frac{4c^2 \times (-1)^n}{n^2\pi^2}}} \quad \& \; b_n = 0$$

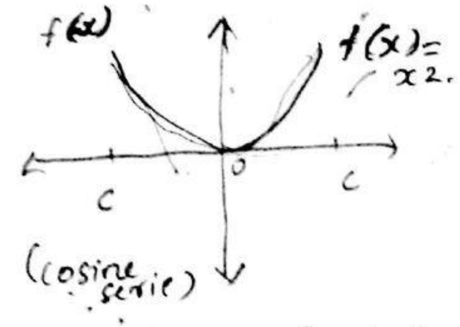
Then the fourier series f(x) corresponding to the required cosine series.

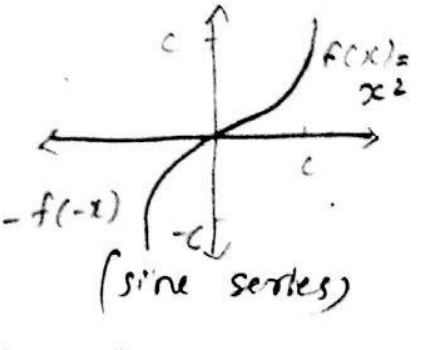
$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right) \qquad \frac{2c^2}{3}\times\frac{1}{2}$$

$$x^2 = \frac{\frac{2c^2}{3}}{2} + \sum_{n=1}^{\infty}\frac{4c^2 \times (-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{c}\right)$$

$$x^2 = \frac{c^2}{3} + \sum_{n=1}^{\infty}\frac{4c^2 \times (-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{c}\right)$$







Half range sine series (odd function).

$a_0 = 0$

$a_n = 0$

$$\frac{-2c^2}{n\pi} \times \cos \ldots$$ 9|3|1

$$b_n = \frac{4c^2}{n^3\pi^3}$$ 3a.

HW

$$b_n = \frac{2}{c}\int_0^c f(x)\sin\left(\frac{n\pi x}{c}\right)dx.$$

$$= \frac{2}{c}\int_0^c \underset{u}{x^2}\cdot \underset{v}{\sin}\left(\frac{n\pi x}{c}\right)dx.$$

$$= \frac{2}{c}\left[\left(x^2 \cdot x - \frac{\cos\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)}\right)_0^c - \left(2x \times - \frac{\sin\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)^2}\right)_0^c + \left(2 \times \frac{\cos\left(\frac{n\pi x}{c}\right)}{\left(\frac{n\pi}{c}\right)^3}\right)\right. \quad 4)$$

$$= \frac{2}{c}\left[-\left(x^2\frac{\cos\left(\frac{n\pi x}{c}\right)}{(n\pi/c)}\right)_0^c + \left(2x\frac{\sin\left(\frac{n\pi x}{c}\right)}{(n\pi/c)^2}\right)_0^c + \left(2\frac{\cos\left(\frac{n\pi x}{c}\right)}{(n\pi/c)^3}\right)_0^c\right.$$ an

$$= \frac{2}{c}\left[\left(-c^2\frac{\cos\left(\frac{n\pi c}{c}\right)}{(n\pi/c)} + 0\right) + \left(2c\frac{\sin\left(\frac{n\pi c}{c}\right)}{(n\pi/c)^2} - 0\right)\right] +$$

$$\left[\frac{2\cos\left(\frac{n\pi c}{c}\right)}{(n\pi/c)^3} - 2\frac{\cos 0}{(n\pi/c)^3}\right]$$

$$= \frac{2}{c}\left(-c^2 \times c\frac{\cos(n\pi)}{(n\pi)} + 2c \times c^2\frac{\sin(n\pi)}{(n\pi)^2} + 2c^3\frac{\cos n\pi}{(n\pi)^3} - 2\frac{c^3}{(n\pi)^3}\right]$$

$$= \frac{2}{\cancel{c}}\left[c^3 \;\; \frac{-\cos n\pi}{n\pi} + \frac{2\sin(n\pi)}{(n\pi)^2} + 2\frac{\cos n\pi}{(n\pi)^3} - \frac{2}{(n\pi)^3}\right]$$

$$= 2c^2\left(\frac{-(-1)^n}{n\pi} + 0 + \frac{2(-1)^n}{(n\pi)^3} - \frac{2}{}\right) = -2c^2\left[\cos(n\pi) + \right.$$

4/3/16

(Half range)

3a. Express $f(x) = x$ as a fourier cosine series $(0 < x < L)$

HW as well as Fourier sine series, $0 < x < 2$.

ans

c

$\underline{c = 2}$

4) Show that the constant 'c' can be expanded in an infinite series $\frac{4c}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right]$,

$0 < x < \pi$ (Half range) $f(x) = 1$

$0 < x < c$

$\boxed{\text{Full} \rightarrow (0, 2\pi) \text{ or } (-\pi, \pi)}$

$\boxed{\text{Half} \rightarrow 0, \pi}$

ans: Here ~~the~~ required series is the Half range sine series for the Function $f(x) = c$ in the interval, $(0, \pi)$

Forwhich $a_0 = 0$, $a_n = 0$.

$$b_n = \frac{2}{\pi}\int_0^c f(x) \sin\left(\frac{n\pi x}{c}\right) dx.$$

$\underline{c = \pi}$.

$$= \frac{2}{\pi}\int_0^{\pi} c.\ \sin\left(\frac{n\pi x}{\pi}\right) dx.$$

$$= \frac{2}{\pi}\int_0^{\pi} c.\ \sin\left(\frac{n\cancel{\pi} x}{\cancel{\pi}}\right) dx.$$

$$= \frac{2}{\pi}\left[c. \frac{-\cos(nx)}{n}\right]_0^{\pi}$$

$$= \cancel{-\frac{2c}{n}\cos}\left[\frac{2}{\pi} \times c\left(\frac{\cos(n\pi)}{n} - \frac{\cos 0}{n}\right)\right]$$

$$= -\frac{2c}{\pi} \times \left(\frac{(-1)^n}{n} - \frac{1}{n}\right)$$

$$= \underline{\underline{-\frac{2c}{\pi} \times \left(\frac{(-1)^n - 1}{n}\right)}} \qquad = \underline{\underline{\frac{2c}{n\pi}\ \cancel{\cos}\left(1 - (-1)^n\right)}}$$

∴ The standard fourier sine series

$$F(x) = \sum_{n=1}^{\infty} bn \ \sin\left(\frac{n\pi x}{C}\right)$$

$$= \sum_{n=1}^{\infty} \frac{2c}{n\pi} \left(1-(-1)^n\right) \ \sin\left(\frac{n\pi x}{\pi}\right)$$

$$= \frac{2c}{\pi} \sum_{n=1}^{\infty} \left[\frac{(1-(-1)^n)}{n}\right] \times \ \sin(nx).$$

$$\frac{2c}{\pi}\left(2 \sin x + 0 + \frac{2}{3} \sin 3x + 0 + \frac{2}{5} \sin 5x\right)$$

$$= \frac{2 \times 2c}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5}\right) ..$$

$$= \frac{4c}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} \cdots\right)$$

5) Find the half range fourier sine series for the function $f(x) = e^{x}$ in the interval

$0 < x < 1$
$0 < x < c$

ans: Here the required ~~half~~ series is a fourier sine series for the function $f(x) = e^{x}$ in the interval $(0,1)$ $(0,c)$

For which,

$a_0 = 0$

$a_n = 0$

$\boxed{\therefore C = 1}$

$$bn = \frac{2}{c} \int_0^c f(x) \sin\left(\frac{n\pi x}{c}\right) dx.$$

$$\therefore bn = \frac{2}{1}\int_0^1 e^x \cdot \sin\left(\frac{n\pi x}{1}\right)dx.$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]}$$

$(a=1 \ , b=n\pi)$

$$\therefore bn = 2\left[\frac{e^x}{1^2+(n\pi)^2} \ 1\sin(n\pi x) - n\pi x\cos(n\pi x)\right]_0^1$$

$$= 2\left[\frac{e^x}{1+n^2\pi^2}\left(\sin(n\pi x) - n\pi\cos(n\pi x)\right)\right]_0^1$$

$$= 2\left[\left(\frac{e^1}{1+n^2\pi^2}(\underset{0}{\sin(n\pi)} - n\pi \times \cos n\pi)\right) - \left(\frac{e^0}{1+n^2\pi^2}\underset{0}{\sin(n\pi x)} - n\pi\cos(n\pi x)\right)\right]$$

$$= 2\left[\frac{-e}{1+n^2\pi^2} \times (-1)^n n\pi - \left(\frac{1}{1+n^2\pi^2} \times -n\pi\right)\right]$$

$$= 2\left(\frac{-e\times(-1)^n \times n\pi}{1+n^2\pi^2} + \frac{n\pi}{1+n^2\pi^2}\right)$$

$$= \frac{2n\pi}{1+n^2\pi^2}\left(1 - e(-1)^n\right)$$

The standard Fourier sine series is

$$F(x) = \sum_{n=1}^{\infty} bn \sin\left(\frac{n\pi x}{l}\right)$$

$$e^x = \sum_{n=1}^{\infty} \frac{2n\pi\left(1-e(-1)^n\right)}{1+n^2\pi^2} \times \sin\left(\frac{n\pi x}{1}\right)$$

6) Find the half range cosine series for the function, $f(x) = (x-1)^2$ in the range $0 \leq x < 1$

HW

ans: $a_0 = 2/3$

$a_n = \cos\frac{n\pi x}{n^2\pi^2} = \frac{(-1)^n - 1}{n\pi^2}$

$b_n = 0$

1) If $f(x) = \begin{cases} x, & \text{for } 0 < x < \pi/2 \\ \pi - x, & \text{for } \pi/2 < x < \pi \end{cases}$

Show that

(i) $f(x) = \frac{4}{\pi}\left[\sin x - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \dots\right]$

(ii) $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \dots\right]$

ans:

(ii) Here the required series is halfrange cosine series for the discontinuous function $f(x)$ in the interval $(0, \pi)$

For which $b_n = 0$.

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx. \qquad (c = \pi)$$

$$= \frac{2}{\pi}\int_0^\pi f(x)\,dx.$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi/2} + \pi(x)_{\pi/2}^{\pi} - \left(\frac{x^2}{2}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2/4}{2} + \pi\left(\pi - \frac{\pi}{2}\right) - \left(\frac{\pi^2}{2} - \frac{(\pi/2)^2}{2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{\pi^2}{8} + \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right)$$

$$= \frac{2}{\pi} \times \frac{\pi^2 \times 2}{8} = \pi \times \frac{4}{8} = \frac{\pi}{2}$$

$$a_n = \frac{2}{c} \int_0^c f(x) \cos\left(\frac{n\pi x}{c}\right) dx$$

$$= \frac{2}{\pi} \cdot \int_0^\pi f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \underset{u \cdot v}{x \cos(nx)}\, dx + \int_{\pi/2}^{\pi} \underset{u \cdot v}{\cancel{x}(\pi - x) \cos(nx)}\, dx\right]$$

$$= \frac{2}{\pi}\left(\left[\left(x \times \frac{\sin(nx)}{n}\right) - \left(1 \times - \frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[\left((\pi - x) \cdot \frac{\sin(nx)}{n}\right) - \left(1 \times - \frac{\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right)$$

$$= \frac{2}{\pi} \times \left[x \frac{\sin(nx)}{n} + \frac{\cos(mx)}{n^2}\right]_0^{\pi/2} + \left[(\pi - x) \frac{\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi} \cdot \left[\left(\frac{\pi}{2} \times \frac{\sin\left(\frac{n\pi}{2}\right)}{n} + \frac{\cos \frac{n\pi}{2}}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right] + \left[\left(0 + \frac{\cos n\pi}{n^2}\right) - \left(\pi - \frac{\pi}{2}\right)\frac{\sin(n\pi/2)}{n} + \frac{\cos(n\pi/2)}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi}{2} \frac{\sin\left(\frac{n\pi}{2}\right)}{n}} + \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2}\right] - \frac{1}{n^2} + \left[\frac{\cancel{\cos}(-1)^n}{n^2} - \cancel{\frac{\pi}{2} \frac{\sin(n\pi/2)}{n}} + \frac{\cos(n\pi/2)}{n^2}\right]$$

$$= \frac{2}{\pi} \cdot 2 \frac{\cos\left(\frac{n\pi}{2}\right)}{n^2} - \frac{1}{n^2} \pm \frac{(-1)^n}{n^2}.$$

$$= \frac{2}{\pi n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{c}\right)$$

$$= \frac{\pi/2}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}\left[2\cos\left(\frac{n\pi}{2}\right) - (-1)^n - 1\right]\left(\cos\frac{n\pi x}{\pi}\right.$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n^2}\left[2\cos\left(\frac{n\pi}{4}\right) - (-1)^n - 1\right]\left(\cos\frac{n\pi}{\pi}\right.$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[\left(\frac{1}{1^2}\, \underset{=0}{2\cos\pi/2} - \underbrace{(-1)^1 - 1}_{=0} \times \cos x\right) + \right.$$
$$\left(\frac{1}{2^2}\, 2\cos(2\,\pi/2) - (-1)^2 - 1\right) \times \cos 2x\Big) + \left(\frac{1}{3^2} \times 2\cos(3\pi/2) - (-1)^3 \ldots \times \cos 3x,\right.$$
$$\tfrac{1}{2} \cdot -1 \quad +1 - 1$$

$$= \frac{\pi}{4} + \frac{2}{\pi}\left[0 + \frac{-1}{2^2}\cos 2x + 0 + 0 - \frac{1}{3^2}\cos 6x - \frac{1}{5^2}\cos 10x - \ldots\right]$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{2^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \ldots\right]$$

Here the required series is halfrange sine series for discontinuous function. f(x) in the intervel, $(0, \pi)$

Forwhich, $a_0 = 0$, $a_n = 0$.

$$b_n = \frac{2}{c}\int_0^{c} f(x)\, \sin\left(\frac{n\pi x}{c}\right) dx. \qquad c = \pi$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\, \sin(nx)\, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \underset{u}{x}\, \underset{v}{\sin(nx)}\, dx + \int_{\pi/2}^{\pi} \underset{u}{(\pi - x)}\, \underset{v}{\sin(nx)}\, dx\right]$$

$$\cancel{= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin(nx)\, dx + \int_{\pi/2}^{\pi} \pi \sin(nx)\, dx - \int_{\pi/2}^{\pi} x \sin\right.}$$

throughc

$$p = \frac{\partial z}{\partial x} = z_x , \quad q = \frac{\partial z}{\partial y} = z_y$$

$$r = \frac{\partial^2 z}{\partial x^2} = z_{xx} , \quad s = \frac{\partial^2 z}{\partial x \partial y} = z_{xy} .$$

$$t = \frac{\partial^2 z}{\partial y^2} = z_{yy} .$$

Order & Degree of a Partial D.E (P.D.E)

The order of a P.D.E, is the order of the higest partial derivative present in that equation.

The degree of a P.D.E, is the degree (power) of the highest order partial derivative present in the equation, when the equation has been made rational & integral as far as the partial derivatives are concerned. For e.g:

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$; order = 1 , degree = 1

$\frac{\partial u}{\partial t} = x \frac{\partial^2 u}{\partial x^2} + \sin x$; order = 2 , degree = 1

$\left\{ \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial u}{\partial y} \right)^2 \right\} y = \left(\frac{\partial z}{\partial y} \right) z$; order = 1 , degree = 2

Number of variable.

The number of variables is the number of independent variables in the P.D.E. For eg:

① $u_t = u_x$ $u \rightarrow$ dependent variable, t, x - independent variables

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$ $> u = f(t, x)$ = 2 variables

② $u_t = \alpha^2 [u_{xx} + u_{yy} + u_{zz}]$

$\frac{\partial u}{\partial t} = \alpha^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$

$u \rightarrow$ dependent
$t, x, y, z \rightarrow$ indepe

No of variables = 4 variables

$u = f(t, x, y, z)$

* linear and non-linear P.D.E

The P.D.E, are classified as being linear & non-linear. A P.D.E, is said to be linear if the dependent variable & its partial derivati are of degree 1. i.e, they appear in linea fashion. and are not multiplied together raised to powers etc. More generally, the general second order P.D.E, in two variab x & y takes the form

$$\boxed{Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu = G} \quad \text{Ⓐ}$$

where A, B, C, D, E, F and G are given functions x & y. otherwise, the P.D.E is said to be non-lin

For eg:

① $u_t = c^2 u_{xx}$. → linear [illegible] indep = (t, x)

② $u_{xx} + u_{yy} + u = \sin x$. → linear

③ $u u_x + y u_y + u = 1$. → non-linear.

④ $u_{xx} + u_{yy} + u^2 = 0$. → non-linear.

- Homogeneous & Non-Homogeneous P.D.E

- In the general P.D.E Ⓐ is homogeneous if $G = 0$ for all x & y in the domain of the equation. otherwise it is said to be non-homogeneous.

Now we are arranging the eqn Ⓐ are

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = G$$

- For homogeneous case $G = 0$ the equation becomes →

$$\left(A \frac{\partial^2}{\partial x^2} + B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} + F\right) u = 0$$

$$Lu = 0 \text{ — } Ⓑ$$

where, 'L' is the differential operator i.e

$$L = A \frac{\partial^2}{\partial x^2} + B \frac{\partial^2}{\partial x \partial y} + C \frac{\partial^2}{\partial y^2} + D \frac{\partial}{\partial x} + E \frac{\partial}{\partial y} + F$$

| 9/16 Friday | The Principle of Super Position |

If $u_1, u_2 \ldots u_n$ be an infinite solutions of the equation Ⓑ then

$$\sum_{n=1}^{\infty} c_n u_n = c_1 u_1 + c_2 u_2 + \cdots\cdots + c_n u_n$$

converges and is differentiable term by term through out some domain of the independent variable, then ~~u=~~

$u = \sum_{n=1}^{\infty} c_n u_n$ is also a solution of equation Ⓑ

P.D.E(s) with constant and variable coefficients.

If the coefficient A, B, C, D, E & F in eqn. Ⓐ a all constants then the equation is said to be a P.D.E with constant coefficients, otherw it is a P.D.E with variable coefficient

Eg:

1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin x$ → P.D.E with constant coefficients (constant=1)

2) $x \frac{du}{dx} + y \frac{du}{dx} = 1.$ → P.D.E with variable coefficients (constants= x

Formation of P.D.E

Now we are going to form a P.D.E

Eliminating arbitary constants
Eliminating arbitary function.

ormation by eliminating arbitary constants.

Suppose we have an equation

$f(x, y, z, a, b) = 0$ —— ⓒ

where, 'a', and 'b' are arbitary constants.
Let us consider z as a dependent variable and 'x' and 'y' are independent variable (ie, Two independent variable).

Now we are eliminating the two arbitary constant in ~~eq~~ 'a', b' in the equation ⓒ by substituting the suitable P. Derivative with respect to x & with respect to y. Then we get after elimination $\phi(x, y, z, p, q) = 0$. —— ⓓ
which is a P.D.E of first order
ie, equation ⓒ & ⓓ show that if the number arbitary constants to be eliminated is equal to the number of independent variable, then we get a first order P.D.E.
otherwise, if the no of arbitary constants ~~to~~ to be eliminated is more than, the no f independent variable, we get a P.D.E of higher order. However, this higher

Note:

If the no of arbitary constants is more than two, then the obtained second order partial derivative to eliminate arbitary constant.

① Find the P.D.E by eliminating the arbitary constants 'a' and 'b' from the following.

(a) $z = (x-a)^2 + (y-b)^2$

(b) $z = ax + by + a^2+b^2$

ans;

(a) x, y - independent variable, a, b - arbitary constant

$$z = (x-a)^2 + (y-b)^2 \text{ ——— } ①$$

Differentiating the ① partially with respect to x

$$\frac{\partial z}{\partial x} = 2(x-a) + 0 \qquad \boxed{\frac{\partial z}{\partial x} = p}$$

$$p = 2(x-a) \Rightarrow x - a = \frac{p}{2} \text{ ——— } ②$$

Differentiating the ① partially w.r. to 'y'.

$$\frac{\partial z}{\partial y} = 0 + 2(y-b)$$

$$q = 2(y-b) \Rightarrow y - b = \frac{q}{2} \text{ ——— } ③$$

Substituting the eqn(s) ② & ③ in ①

$$\therefore z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$z = \frac{p^2+q^2}{4}$$

$\Rightarrow 4z = p^2+q^2$ which is the required P.D.E with first order

(ii) $z = ax + by + a^2 + b^2$ —— ①

ans Differentiating the eqn ① partially w.r.to x.

$$\frac{\partial z}{\partial x} = a \longrightarrow$$

$$\Rightarrow p = a \text{ —— } ②$$

Differentiating the eqn ① partially w.r.to 'y'

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \text{ —— } ③$$

Substituting the eqn(s) ② & ③ in ①.

$\therefore z = px + \underline{qy +} p^2 + q^2$, which is the required D.E with first order. PDE

② Find the D.E for all spheres, of fixed radius having their centres in the XY plane

ans: The equation of the sphere with centre at $(h, k, 0)$ in the xy-plane & radius 'r' is given by.

$$(x-h)^2 + (y-k)^2 + z^2 = r^2 \longrightarrow ①$$

Differentiating the ① partially w.r. to x.

$$2(x-h) + 0 + 2z\frac{\partial z}{\partial x} = 0$$

$$2\left[(x-h) + z\frac{\partial z}{\partial x}\right] = 0$$

$$(x-h) + zp = 0$$

$$x-h = -zp \quad \text{——} ②$$

Differentiating ① partiall w. r. to y:

$$0 + 2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

$$2\left[(y-k) + z\frac{\partial z}{\partial y}\right] = 0$$

$$y-k + zq = 0$$

$$y-k = -zq \quad \text{——} ③$$

Substituting ② & ③ in ①

$$\Rightarrow \quad (-zp)^2 + (-zq)^2 + z^2 = r^2$$

$$z^2p^2 + z^2q^2 + z^2 = r^2$$

$$\underline{z^2(p^2+q^2+1)} = r^2$$, which is the required First order PDE

15/3/16
Tuesday

Formation of PDE by eliminating the arbitary fuction.

1. Q. Form the PDE by eliminating the arbitary function.

① $z = f(x^2-y^2)$

$z = f(x^2-y^2)$ — ①

Differentiating the eqn ① partially w. r. to x.

$$\frac{\partial z}{\partial x} = f'(x^2-y^2)\times 2x$$

$$\Rightarrow P = 2x\times f'(x^2-y^2) \text{ — ②}$$

Differentiating the eqn ① partially w. r to y.

$$\frac{\partial z}{\partial y} = f'(x^2-y^2)\times -2y.$$

$$q = -2y\times f'(x^2-y^2) \text{ — ③}$$

Dividing

② / ③ $\Rightarrow$ $\frac{P}{q} = \frac{-x}{y}$ $\Rightarrow Py = -qx$

$\Rightarrow \underline{py+qx=0}.$

which is a first order P.D.E.

$$z = f\left(\frac{x.y}{z}\right)$$

$$z = f\left(\frac{xy}{z}\right) \text{ — ①}$$

Differentiating eqn ① partially with respect to x

quotient rule

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\times\frac{\partial}{\partial x}\left(\frac{xy}{z}\right)$$

$$p = f'\left(\frac{xy}{z}\right)\left[\frac{z\times\frac{\partial}{\partial x}(xy) - xy\times\frac{\partial z}{\partial x}}{z^2}\right] \qquad \boxed{\frac{\partial z}{\partial x} = p}$$

$$p = f'\left(\frac{xy}{z}\right)\left[\frac{z\times(y\times 1) - xy\times p}{z^2}\right]$$

$$p = y\times f'\left(\frac{xy}{z}\right)\times\left(\frac{z-xp}{z^2}\right) \text{ — ②}$$

Similarly Differentiating eqn ① partially w.r.to y.

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\frac{\partial}{\partial y}\left(\frac{xy}{z}\right)$$

$$q = f'\left(\frac{xy}{z}\right)\left[\frac{z \times \frac{\partial}{\partial y}(xy) - (xy) \times \frac{\partial z}{\partial y}}{z^2}\right]$$

$$q = f'\left(\frac{xy}{z}\right)\left[\frac{z \times (x \times 1) - xy \times q}{z^2}\right] \qquad \boxed{\frac{\partial z}{\partial y} = q}$$

$$q = x \times f'\left(\frac{xy}{z}\right)\left[\frac{z - yq}{z^2}\right] \text{ — ③}$$

$$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{y(z - xp)}{x(z - yq)} \Rightarrow \frac{p}{q} = \frac{yz - xyp}{xz - xyq}$$

~~$p(x(z - yq)) = q(y(z - xp))$~~

~~$px(pz - pqy) = qy(qz - pzx)$~~

~~$p^2xzq - p^2xy$~~ $= p(xz - xyq) = q(yz - xyp)$

$pxz - xypq - qyz + xypq = 0$

$(px - qy)z = 0.$

$\Rightarrow \therefore px - qy = 0$

⑧ $z = f(x + iy) + g(x - iy)$, $i = \sqrt{-1}$

ans: $z = f(x + iy) + g(x - iy)$ — ①

Differentiating the ① partially w.r.to x.

$$\frac{\partial z}{\partial x} = f'(x + iy) \times 1 + g'(x - iy) \times 1 \qquad \partial/\partial x = 1$$

$$p = f'(x + iy) + g'(x - iy) \text{ — ②}$$

Differentiating the ② partially w.r.to y

$$\frac{\partial z}{\partial y} = f'(x + iy) \times i + g'(x - iy) \times -i.$$

$$q = i\left[f'(x + iy) - g'(x - iy)\right] \text{ — ③}$$

Differentiating the ② again partially w.r.to x.

$$\boxed{\frac{\partial^2 z}{\partial x^2} = r} \quad \boxed{\frac{\partial^2 z}{\partial x \partial y} = s} \quad \boxed{\frac{\partial^2 z}{\partial y^2} = t}$$

i.e. $r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy)\times 1 + g''(x-iy)\times 1$

$$r = f''(x+iy) + g''(x-iy) \text{ —— ④}$$

Differentiating the ③ $\overset{\text{partially}}{\text{with}}$ respect to y.

$$t = \frac{\partial^2 z}{\partial y^2} = i[f''(x+iy)\times i - g''(x-iy)\times -i]$$

$$t = i^2 [f''(x+iy) + g''(x-iy)] \quad \boxed{i^2 = -1}$$

$$t = -[f''(x+iy) + g''(x-iy)] \text{ —— ⑤}$$

∴ from the eqn ④ and ⑤, we have

$$r = -t, \quad \text{i.e.} \quad r + t = 0,$$

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$ which is a second order P.D.E

Form the P.D.E from the equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, by eliminating the arbitary constants.

(i.e no. of independent variable to be eliminated = arbitary constants)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \text{ —— ①}$$

Differentiating ① $\overset{\text{partially}}{\text{w. r.}}$ to x.

$$\frac{2x}{a^2} + 0 + \frac{1}{c^2}\left(2z\frac{\partial z}{\partial x}\right) = 0 \Rightarrow \frac{2x}{a^2} + \frac{1}{c^2}\left(2z\cdot\frac{\partial z}{\partial x}\right) = 0 \text{ —— ②}$$

Diff. the ② partially w. r. to y

$$0 + \frac{2}{c^2}\left(\frac{\partial}{\partial y}\left(z\cdot\frac{\partial z}{\partial x}\right)\right) = 0$$

In this problem the number of arbitary constants to be eliminated = 3, which is greater than the number of independent variable which is equal to 2 (x, y) (product rule) of Differentiation

ie; $\frac{2}{c^2}\left[z\cdot\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)+\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right]=0$

ie, $\frac{2}{c^2}\left[z\frac{\partial^2 z}{\partial y\,\partial x}+\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}\right]=0$

Since, $\frac{2}{c^2}\neq 0$ (c is arbitary constant $c\neq 0$)

$$\Rightarrow\quad z\frac{\partial^2 z}{\partial y\,\partial x}+\underline{\underline{\frac{\partial z}{\partial x}}}\frac{\partial z}{\partial y}=0.$$

which is a <u>p.D.E of</u> second order.

3) Form the P.D.E by eliminating the arbitary functions in, $z=yf(x)+x\cdot g(y)$

ans; $z=y\,f(x)+x\cdot g(y)$ —— ①

Differentiating the ① partially w.r.to x.

$$\frac{\partial z}{\partial x}=y\cdot f'(x)\times 1+g(y)\times 1$$

$$p=y\,f'(x)+g(y) \text{ —— } ②$$

Differentiating the ① partially w.r to y!

ie $\frac{\partial z}{\partial y}=f(x)\times 1+x\times g'(y)\times 1$

$$q=f(x)+x\,g'(y) \text{ —— } ③$$

Differentiating the ② partially w.r.to y.

$$\frac{\partial^2 z}{\partial x\,\partial y}=S=f'(x)\times 1+g'(y) \text{ —— } ④$$

Eliminating the $f'(x)$ and $g'(x)$ from in eqn ④ with the help of the eqn(s) ② & ③

$$f'(x) = \frac{p - g(y)}{y} \quad , \quad g'(y) = \frac{q - f(x)}{x}$$

$$s = \left[\frac{p - g(y)}{y}\right] + \left[\frac{q - f(x)}{x}\right]$$

$$s = \frac{x(p - g(y)) + y(q - f(x))}{xy}$$

$$s = \frac{xp - x \times g(y) + qy - y \times f(x)}{xy}$$

from eqn ①

$$z = y \times f(x) + x \times g(y).$$

$$xy \cdot s = px + qy - xg(y) - yf(x)$$

$$xys = \underline{\underline{px + qy - z}}$$, which is a P.D.E of second order.

Formation of PDE by eliminating the arbitary function 'φ' from the given equation $\phi(u,v) = 0$, where u & v are the function of x, y and z.

* * *

Here u, v are the two functions containing the '3' independent variables x, y & z. and 'φ' is an arbitary function.

By eliminating the arbitary function 'φ' from the relation $\phi(u,v) = 0$ we get

a first order P.D.E in p, q and ...
is the form '$P_p + Q_q = R$' where P, Q, R
are the functions of the 3 independent
variables x, y & z where ~~P=~~

.yz.
zx,
xy.

$$\boxed{\begin{aligned} P &= \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} \\ Q &= \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} \\ R &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x} \end{aligned}}$$

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wednesday

③ Q. Find the D.E of all planes which are ~~not~~ at a constant distance from the origin.

ans. The eqn of all planes which are at a constant distance 'k' units from origin takes the form, $lx + my + nz = k$ — ①

where $l^2 + m^2 + n^2 = 1^2$

Differentiating ① with respect to x & y

$l + n\frac{\partial z}{\partial x} = 0 \Rightarrow l + np = 0$ — ②

n-[illegible]

$m + n\frac{\partial z}{\partial y} = 0 \Rightarrow m + nq = 0$ — ③

eliminating l, m, and n using ①, ② & ③

① becomes $(-np)x + (-nq)y + nz = k\sqrt{l^2+m^2+n^2}$

$n(-px-qy+z) = k\sqrt{(-np)^2+(-nq)^2+n^2}$

$n(-px-qy+z) = kn\sqrt{p^2+q^2+1}$

$z = px+qy+k\sqrt{p^2+q^2+1}$, which is partial D.E of first order.

Note: In all of the above eg's the number of arbitary constants is equal to the number of dependent variables. Hence the partial D.E(s) are of first order.

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Continues

Note.

① If the given relation b/w x, y & z contains two arbitary functions & then in general there elimination will give a second order P.D.E

② If the arbitary function are more than 2, then we need the higher order P.Derivatives for elimination.

③ The relation $\phi(u,v) = 0$ involving the arbitary function ϕ can be written in another form like $u = \phi(v)$ or $v = \phi(u)$

after elimination of the arbitary function ϕ we get a first order in p & q is of the form $Pp + Qq = R$

where $$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$\therefore P = 2y \times 2z - 2z \times (-x)$$

$$P = \underline{\underline{4yz + 2xz}}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

$$= 2z \times (-y) - 2x \cdot 2z$$

$$= -2yz - 4xz$$

$$Q = \underline{\underline{-(2yz + 4xz)}}$$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$= 2x \cdot \times (-x) - 2y \times (-y)$$

$$= -2x^2 + 2y^2$$

$$R = \underline{\underline{2(y^2 - x^2)}}$$

The required P.D.E is of the form $p = \frac{\partial^2 x}{\partial z^2}$

$$(4yz + 2xz)p - (2yz + 4xz)q = 2(y^2 - x^2)$$

$$(2yz + xz)p - (yz + 2xz)q = y^2 - x^2$$ which is of orde

⑫ Form a p.D.E by eliminating the arbitary function ϕ from $x.y.z = \phi(x+y+z)$

ans: The given relation is of the form,

$$u = \phi(v)$$

where $u = x.y.z \quad , \quad v = x+y+z.$

after elimination the arbitary function ϕ from the given relation we get a first order p.D.E is of the form $Pp + Qq = R$.

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$= xz \times 1 - xy \times 1.$$

$$P = xz - xy \quad \Rightarrow \quad P = \underline{\underline{x(z-y)}}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}.$$

$$= xy \times 1 - yz \times 1.$$

$$Q = xy - yz \quad \Rightarrow \quad Q = \underline{\underline{y(x-z)}}$$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}.$$

$$= yz \times 1 - xz \times 1.$$

$$R = yz - xz \quad \rightarrow R = \underline{\underline{z(y-x)}}$$

After elimination,
∴ the required p.D.E is of the form,

$x(z-y)p + y(x-z)q = z(y-x)$

which is a p.D.E of first order.

Solution of first order P.D.E's

Before going to solve a P.D.E we have to discuss about some kinds of solutions of P.DE

① Complete Integral.

We have seen that the elimination of the arbitary constant from the relation ~~f(x,y,z)~~

$f(x,y,z,a,b) = 0$. —— Ⓐ

gives the following P.D.E

$\phi(x,y,z,p,q) = 0$ —— Ⓑ

If the equation Ⓐ is derived from the equation Ⓑ by any method. Then the eqnⒶ which contains the arbitary constant that are equal to the number of indpendent variable is called The complete ~~the~~ integral of the equation Ⓑ

② Particular Integral

A particular integral of the eqnⒷ is obtained b giving particular values to the constants a b in the complete integral.

③ General Integral

The elimination of the arbitary function 'φ' from the relation $\phi(u,v)=0$, where u and v are the function of the 3 independent variables $x, y, \& z$ gives a linear P.D.E of the form, $P_p + Q_q = R$, If we say that $\phi(u,v)=0$ is the solution of the P.D.E

$P_p + Q_q = R$, then it is know as the general integral.

④ Singular Integral

The envelope of the surface of the equation $f(x,y,z,a,b)=0$ —Ⓐ is obtained by eliminating the arbitary constants a & b from the eqn ~~Res~~ $f=0$, $\frac{\partial f}{\partial a}=0$ & $\frac{\partial f}{\partial b}=0$.

The above eqn gives a relation b/w $x, y, \& z$ which is called the singular integral.

Note:

Let the complete integral be $f(x,y,z,a,b)=0$ and let us assume that one of the arbitary Constant be a function of other say $b=\phi(a)$. then our eqn becomes $f(x,y,z,a,\phi(a))=0$

The general integral is obtained by eliminating the arbitary constant 'a' from $f(x,y,z,a,\phi(a))=0$ & $\frac{\partial f}{\partial a}=0$. This eqn gives the curve of intersection of two consecutive surfaces given by the relation $f(x,y,z,a,\phi(a))=0$ This curve is known as the envelope of the family

Lagrange's Method

consider a first order P.D.E $Pp+Qq=R$ which is linear in p and q were P, Q, & R are junction of x, y, z. The procedure for the large method is used for finding u & v as the solution of the above P.D.E

Theorem.

The general solution of the linear P.D.E $Pp+Qq=R$ is $\phi(u,v)=0$, where ϕ is an arbitary function and u, v are the function of 3 independent variable x, y & z ie; $u(x,y,z)=c$ & $u(x,y,c)$ are the two independent solution of Lagranges auxillary equation $\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}}$

Working Rule

To obtain the solutions of PDE, $Pp+Qq=R$ from the Lagranges auxillary equation.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

. Find two independent solution of these auxillary equation as $u(x,y,z)=c$ and $v(x,y,z)=c_2$ and the general solution of P.DE $Pp+Qq=R$, is given by one of the following forms $\phi(u,v)=0$ $\boxed{\text{or}}$

$u=\phi(v)$, $v=\phi(u)$.

In order ~~of~~ to obtain the two independent solution we use the following methods.

(i) Method of grouping (Rule 1)

Suppose that one of the variable is either object or cancel out from any pair of the fractions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, and then a solution can be obtained by using usual methods. To same procedure is repealed with another pair of fraction of the above equation for II^{nd} independent solutions.

(ii) Methods of multipliers (Rule 2)

If l, m, n are the three multiplies then by a were none principle of algebra each fraction of eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ will be equal to

$$\frac{ldx + mdy + ndz}{l + m + n}$$

If it is possible to choose l, m, n such that $lp + mQ + nR = 0$. Then $ldx + mdy + ndz = 0$. which can be integrated to give $u(x,y,z) = c$ The method may be repeated to get another independent solution $v(x,y,z) = c_2$, The multiplier may be functions of x, y, z)/constants Some times, it is required to find only one solution using multipliers.

* Note - 1

If one solution of the eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is determined by rule-1 & suppose that we can't determine the 2nd solution by rule 1 or rule 2 then first solution known to us is used to find the another independent solutions.

* Note - 2

If one solution is found that $g(x,y) = c_1$ then the constant c_1 should be used in the place of $g(x,y)$ throughout the process of calculating the 2nd solution and finally when 2nd solution is found involving c_1 then c_1 should again be replaced by $g(x,y)$.

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Monday

① Solve $y^2zp + x^2zq = xy^2$ · T

The given eqn is a Lagrange's D.E of the form $Pp + Qq = R$

and the Lagrange A.E is $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$

$P = y^2z, \quad Q = x^2z, \quad R = xy^2$

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2} \quad (A.E) \quad \text{——} ①$$

taking the first two ratio, we get

$$\frac{dx}{y^2\cancel{z}} = \frac{dy}{x^2\cancel{z}}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$

$$x^2\,dx = y^2\,dy$$

then on integration we get

$$\int x^2\,dx = \int y^2\,dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\frac{x^3 - y^3}{3} = c \Rightarrow \therefore x^3 - y^3 = 3c.$$

$$c_1 = 3c \qquad \therefore \underline{x^3 - y^3 = c_1}$$

which is our first independent solution of the form $u(x, y, z) = c_1$

For getting the second independent solution we have to take the first & third ratio

i.e., $\dfrac{dx}{y^2 z} = \dfrac{dz}{x y^2}$

$$\frac{dx}{z} = \frac{dz}{x} \Rightarrow x\,dx = z\,dz.$$

Then on integrations we get

$$\int x\,dx = \int z\,dz.$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c$$

$$\frac{x^2 - z^2}{2} = c \Rightarrow x^2 - z^2 = 2c.$$

$$2c = c_2 \Rightarrow \underline{x^2 - z^2 = c_2}.$$

which is our second independent solution of the form $v(x, y, z) = c_2$

Then the general solution $\phi(u, V) = 0$.

$$\text{ie, } \phi\left(x^3 - y^3,\ x^2 - z^2\right) = 0$$

② Solve

$$p \tan x + q \tan y = \tan z$$

ans: The given eqn is a Lagrange D.E is of the form

$$Pp + Qq = R.$$

Then the Lagrange A.E are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\text{ie, } \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad \text{—} ①$$

For getting the first indepent solution We are taking the first 2 Ratio

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Then on integration, $\int \frac{dx}{\tan x} = \int \frac{dy}{\tan y}$

$$\text{ie, } \int \cot x \, dx = \int \cot y \, dy$$

$$\text{ie, } \log \sin x = \log \sin y + \log C_1$$

$$\text{ie, } \log \sin x = \log (C_1 \sin y)$$

$$\sin x = C_1 \sin y$$

$\int \cot x \, dx =$

$\int \frac{\cos x}{\sin x} dx$

$\int \frac{du}{u}$ $u = \sin x$

$du = \cos x$

$= \log(u) + c$

$\frac{\sin x}{\sin y} = c_1$ which is our first independent solution is of the form $u(x, y, z) = c_1$

forgetting the 2nd independent variable, we are taking 2nd & 3rd ratio

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Then on integration $\int \frac{dy}{\tan y} = \int \frac{dz}{\tan z}$

$$\int \cot y \, dy = \int \cot z \, dz$$

$\log a + \log b = \log a \cdot b$

$$\log \sin y = \log \sin z + \log c_2$$

$$\log \sin y = \log (c_2 \sin z)$$

$$\sin y = c_2 \sin z$$

$\frac{\sin y}{\sin z} = c_2$, which is our 2nd independent solution is of the form $V(x, y, z) = c_2$.

Then our general solution $\phi(u, v) = 0$.

i.e, $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

HW

③ Solve $xp + yq = 3z$.

HW

④ Solve $x^2 p + y^2 q = z^2$

⑤ Solve. $p-q = \log(x+y)$

ans: The given eqn is a lagrang D.E is of the form

$Pp + Qq = R$

The Lagrange A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)} \quad \text{——①}$$

for getting the 1st independent solution we are grouping the first & second Ratio

ie $\frac{dx}{1} = \frac{dy}{-1} \Rightarrow \underline{dx} = -dy$

Then on integration we get,

$$\int dx = \int dy$$

$$x = -y + c_1$$

$$x+\underline{\underline{y}} = c_1 \quad \text{——②} \quad \text{~~which~~ is first eqn } u(x,y,z)=c$$

For getting the second independent solution we are grouping the first & third ratio

ie, $\frac{dx}{1} = \frac{dz}{\log(x+y)}$

from ② $\Rightarrow$ $dx = \frac{dz}{\log c_1}$

$$\log c_1 \, dx = dz$$

Then on integration, we get

$$\int \log c_1 \, dx = \int dz$$

$$x \log c_1 = z + c_2$$

ie $x\log(x+y) - z = c_2$ —— ⑧

which is our 2nd independent solution of the form $\psi(x, y, z) = c_2$

∴ The general solution is of the form $\phi(u, v) = 0$

ie $\phi(x+y,\ \underline{x\log(x+y) - z}) = 0.$

⑥ Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

The Given eqn is a Lagrange D.E is of the form

$Pp + Qq = R$

ie A.E $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

ie $$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad —— ①$$

By method of multipliers

$$\boxed{\begin{array}{c} l, m, n \\ \dfrac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \end{array}} \boxed{\frac{1}{x}, \frac{1}{y}, \frac{1}{z}}$$

~~Taking~~ FOR getting the $\overset{two}{\underline{2}}$ independent solution we can't apply the method of grouping, so here we are choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are Multipliers. Then we get

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{}$$

denominator

$$x(y-z)+y(z-x)+z(x-y)$$

$$xy-xz+yz-xy+xz-yz=0$$

ie,

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

considering this with any fraction in ①

ie, $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

Then on integration

$$\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$$

$$\log x+\log y+\log z=\log c_1$$

$$\log(xyz)=\log c_1$$

$xyz=c_1$ This is our one independe solution of the form $u(x,y,z)=c_1$

ans:

③ $x_p+y_q=3z$

The given equation is a Lagrange D.E of the form $P_p+Q_q=R$

Then the Lagrange A.E is,

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$P=x$, $Q=y$, $R=3z$

∴ $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$

For getting the first independent solution we are taking the first 2 ratio

$$\frac{dx}{x} = \frac{dy}{y}$$

then on integration, $\int \frac{dx}{x} = \int \frac{dy}{y}$

$$\text{ie, } \log x = \log y + \log c_1.$$

$$\log x = \log(c_1 \times y)$$

$$x = c_1 y$$

⇒ $\frac{x}{y} = c_1$ which is our first independent solution is of the form $u(x,y,z) = c_1$.

For getting the 2nd independent variable we are taking 2nd & 3rd ratio

$$\frac{dy}{y} = \frac{dz}{3z}$$

Then on integration, $\int \frac{dy}{y} = \int \frac{dz}{3z}$.

$$\log y = \frac{1}{3}\log z + \log c_2$$

$$\log y = \frac{1}{3}\log(c_2 \times z)$$

$$y = \frac{1}{3} \times (c_2 \times z) \Rightarrow \frac{3y}{z} = c_2$$

which is our 2nd independent solution is of the form $v(x,y,z) = c_2$

Then the general solution $\phi(u, v) = 0$

i.e, $\phi\left(\frac{x}{y}, \frac{3y}{z}\right) = 0$

④ $x^2 p + y^2 q = z^2$

ans; The given eqn is Lagrage D.E is of the form $Pp + Qq = R$.

Then the Lagrange A.E is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \quad P = x^2$$

$$Q = y^2$$

$$R = z^2$$

$$\therefore \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}.$$

Taking the first two ratio

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$u = x^2$
$du = 2x\,d$
$\frac{du}{2x}$

Then on integration, $\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$

i.e, $\int x^{-2} dx = \int y^{-2} dy$

$$\frac{x^{-2+1}}{-2+1} = \frac{y^{-2+1}}{-2+1} + C_1$$

$$-x^{-1} = -y^{-1}$$

$$-\frac{1}{x} = -\frac{1}{y} + C_1$$

$$-\frac{1}{x}+\frac{1}{y}=c_1 \Rightarrow \frac{1}{y}-\frac{1}{x}=c_1$$

which is our first independent solution of the form $u(x,y,z)=c_1$

For getting the 2nd Independent solution Taking 2nd & 3rd Ratios

$$\frac{dy}{y^2}=\frac{dz}{z^2}$$

Then on integration $\int\frac{dy}{y^2}=\int\frac{dz}{z^2}$

$$\int y^{-2}dy=\int z^{-2}dz$$

$$\frac{y^{-2+1}}{-2+1}=\frac{z^{-2+1}}{-2+1}+c_2$$

$$-\frac{1}{y}=-\frac{1}{z}+c_2$$

$$\therefore -\frac{1}{y}+\frac{1}{z}=c_2 \Rightarrow \frac{1}{z}-\frac{1}{y}=c_2$$

which is our second ind independent solution of the form $v(x,y,z)=c_2$

Then the general solution $\phi(u,v)=0$

$$\text{i.e., } \phi\left(\left(\frac{1}{y}-\frac{1}{x}\right),\left(\frac{1}{z}-\frac{1}{y}\right)\right)=0$$

⑥ Que. continue.

For getting the 2nd independent solution we are choosing $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$ are multiplies then each fraction in ① will be equal to

$$\frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{(y-z)+(z-x)+(x-y)} = \frac{\frac{dx}{x^2}+\frac{dy}{y2}+\frac{dz}{z^2}}{0}$$

taking the above fraction with any fract. of ① we get,

$$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2} = 0$$

Then on integration we get,

$$\int\frac{dx}{x^2} + \int\frac{dy}{y2} + \int\frac{dz}{z^2} = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = C_2$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = -C_2 \qquad \boxed{-c_2 = C_2}$$

$$\therefore \underline{\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = C_2}$$

which is the 2nd independent solution of the form $v(x,y,z) = C_2$

Then the general solution is

$$\phi\left(x,y,z,\ \underline{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\right) = 0$$

⑦ Solve $(y^2+z^2)p - xyq + xz = 0$

ans: which is of the form $Pp + Qq = R$.

$$(y^2+z^2)p - xyq = -(xz) \qquad \boxed{\begin{array}{l} P = y^2+z^2, \\ Q = -(xy) \\ R = -(xz) \end{array}}$$

Then the Lagrange A·E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e, $\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$

For getting the first independent solution we are taking the second & third fraction

$$\frac{dy}{-xy} = \frac{dz}{-xz}$$

i.e, $\frac{dy}{y} = \frac{dz}{z}$

Then on integration we get

$$\int\frac{dy}{y} = \int\frac{dz}{z}, \quad \log y = \log z + \log c_1$$

$$\therefore \log y = \log(c_1 \cdot z)$$

$$\therefore y = c_1 \times z$$

$$\therefore \frac{y}{z} = c_1 \text{ --- } ②$$

which is the first independent solution of the form $u(x,y,z) = c_1$.

- For getting the 2nd independent solution, we are choosing x, y, z are multipliers Then each fraction in ① will be equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) + y(-xy) + z(-xz)}$$

$$= x\,dx + y\,dy + z\,dz$$

i.e $x\,dx + y\,dy + z\,dz = 0$

Then on integration $\int x\,dx + \underline{\int y\,dy} + \int z\,dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$$

$$\frac{x^2+y^2+z^2}{2} = C_2 \Rightarrow x^2+y^2+z^2 = 2C_2$$

$$\underline{\underline{x^2+y^2+z^2}} = C_2. \qquad \boxed{2C_2 = C_2}$$

which is our IInd independent solution of the form $V(x,y,z) = C_2$

G.S is $\phi(u, v) = 0$

$$\phi\left(\frac{y}{z},\ x^2+y^2+z^2\right) = 0.$$

⑧ Solve $xy\,dx + y^2 dy = zxy - 2x^2$.

which is a Lagrange D.E of the form $Pp + Qq = R$.

& the Lagrage A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e, $$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2} \quad \text{—①}$$

For getting the first independent solution we are choosing the first two fraction we get

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Then on integration we get $\int \frac{dx}{x} = \int \frac{dy}{y}$

$$\log x = \log y + \log c_1$$

$$\log x = \log(y \times c_1)$$

$$\therefore x = y \times c_1 \Rightarrow x = c_1 y$$

$$\therefore \frac{x}{y} = c_1 \quad \text{——} ②$$

which is first independent solution of the form ab$(x, y, z) = C_1$.

For getting the 2nd independ solution we are using the first independent solution, $\underline{x = c_1 y}$

ie For that we are taking the 2nd & 3rd fraction

$$\frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$$

$$\frac{dy}{y^2} = \frac{dz}{zc_1y^2 - 2c_1^2y^2} \quad \text{by } ②$$

$$\frac{dy}{y^2} = \frac{dz}{y^2(zc_1 - 2c_1^2)}$$

$$dy = \frac{dz}{zc_1 - 2c_1^2}$$

$$dy = \frac{dz}{c_1(z - 2c_1)}$$

$$c_1\, dy = \frac{dz}{z - 2c_1}$$

Then on integration we get $\int c_1\, dy = \int \frac{dz}{z - 2c_1}$

$$c_1 y = \frac{\cancel{c_1}}{-\cancel{2c_1}} \log(z - 2c_1) + \log c_2$$

$$\boxed{x = c_1 y}$$

$$\boxed{c_1 = \frac{x}{y}}$$

$$x = \log (z - 2c_1) + \log c_2$$

$$x = \log \left(z - \frac{2x}{y}\right) \cancel{+ c_2} \log c_2 .$$

$$x = \log \left(\frac{yz - 2x}{y}\right) + \log c_2$$

$$\boxed{\begin{array}{l} e^x = \frac{yz - 2x}{y} \\ ye^x = (yz - 2x) \\ \frac{ye^x}{yz - 2x} = c_2 \end{array}}$$

$$\frac{ye^x}{yz - 2x} = c_2$$

which is our ~~II~~ independent solution $V(x, y, z) = c_2$

i.e our G.S $\phi(u, v) = 0$

$$\phi\left(\frac{x}{y}, \frac{ye^x}{yz - 2x}\right) = 0$$

ⓠ Solve $p - 2q = 3x^2 \sin(y + 2x)$

ans. The give eqn is a Lagrange D.E is of the form $Pp + Qq = R$.

Then the Lagrange A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y+2x)}$ — ①

for getting the 1st independent solution we are taking the first two fraction

$$\frac{dx}{1} = \frac{dy}{-2}$$

i.e $2dx = -dy$

then on integration, we get $\int 2\,dx = \int -dy$

$$2x = -y + c_1$$

$$\underline{2x + y = c_1} \quad \text{——②}$$

which is our first independent solution of the form $u(x,y,z) = c_1$.

for getting the 2nd independent solution we have to make use of 1st independent solution, for that we use taking the Ist and 3rd fraction

by —②
$\boxed{y + 2x = c_1}$

$$\frac{dx}{1} = \frac{dz}{3x^2 \sin(y+2x)}$$

$$dx = \frac{dz}{3x^2 \sin c_1}$$

~~Th~~

$$3x^2 \sin c_1 \; dx = dz.$$

Then on integration we get; $\int 3x^2 \sin c_1 \, dx = \int dz$

$$3 \sin c_1 \frac{x^3}{3} = z + c_2$$

$$x^3 \sin c_1 = z + c_2$$

$$x^3 \sin c_1 - z = c_2$$

$$x^3 \sin(y+2x) - z = c_2 \quad \text{——③}$$

which our 2nd solution of the form

$v(x,y,z) = c_2$,

Then the G.S, $\phi(u,v) = 0$ ie $\phi(2x+y, \; x^3 \sin(y+2x) - z) = 0$

are taking the third & last fraction

$$\frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x^2(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2}$$

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

Then on integration. we get,

$$\int \frac{dz}{z} = \int \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$u = x^2+y^2+z^2$
$du = 2x\,dx + 2y\,dy + 2z\,d$

$\int \frac{du}{u} = \log(u)$

$$\log z = \log(x^2+y^2+z^2) + \log c_2$$

$$\therefore \log z = \log((x^2+y^2+z^2) \times c_2)$$

$$\therefore \frac{z}{x^2+y^2+z^2} = c_2 \quad \text{———} ③$$

which is our 2nd independent solution. then the G.S is of the form $\phi(u, v) = 0$

$$\text{ie, } \therefore \phi\left(\frac{y}{z}, \frac{z}{x^2+y^2+z^2}\right) = 0$$

solve $(y+z)p + (z+x)q = x+y$

Thes given eqn is a first order Lagrange P.E of the form $Pp + Qq = R$

Then the Lagrange A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{ie } \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{———} ①$$

For getting two solutions the method of grouping & multipliers are difficult in some way. So, we are setting some other fractions to ①

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{(y+z)-(z+x)} = \frac{dy-dz}{(z+x)-(x+y)} = \frac{dx+dy+dz}{y+z+z+x+x+y}$$

$$\Rightarrow \underset{①}{\frac{dx}{y+z}} = \underset{②}{\frac{dy}{z+x}} = \underset{③}{\frac{dz}{x+y}} = \underset{④}{\frac{dx-dy}{y-x}} = \underset{⑤}{\frac{dy-dz}{z-y}} = \underset{⑥}{\frac{dx+dy+dz}{2(x+y+z)}}$$

For getting the first independent solution we are taking in 4th & 5th ratio

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\frac{dx-dy}{-(x-y)} = \frac{dy-dz}{-(y-z)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$, then on integration

$$\int\frac{dx-dy}{x-y} = \int\frac{dy-dz}{y-z}$$

$u = x-y$
$du = dx-dy$
$\int\frac{du}{u} = \log|u|$

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log(x-y) = \log((y-z)\times c_1)$$

$$\therefore x-y = y-z \times c_1 \Rightarrow \frac{x-y}{y-z} = c_1 \text{ —— } ②$$

which is the 1st independent solution of the e for u(x,y getting the 2nd independent solution we are taking the fourth and sixth ration (4^{th} & 6^{th})

$$\frac{dx-dy}{y-x} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{dx-dy}{-(x-y)} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$-2\times\left(\frac{dx-dy}{-(x-y)}\right) = \frac{dx+dy+dz}{x+y+z.}$$

$u = x+y+z$

$du = dx+dy+dz$

$\int \frac{dy}{u} = \log u$

$u = x-y$

$du = dx-dy$

$$2\log(x-y) = \log(x+y+z) + \log C_2$$

$$\therefore \log(x-y)^{-2} = \log((x+y+z)\times C_2)$$

$$(x-y)^{-2} = ((x+y+z)\times C_2)$$

$$\frac{1}{(x-y)^2(x+y+z)} = C_2. \quad \text{—③}$$

Then the G.S is of the form $\phi(u,v) = 0$

i.e, $\phi\left(\frac{x-y}{y-z}, \frac{1}{(x-y)^2(x+y+z)}\right) = 0$

Generalised form of Lagrange's Method.

Let the n^{th} order Lagrange equation is of the form $P_1p_1 + P_2p_2 \ldots\ldots + P_np_n = R$

when $p_1 = \frac{\partial z}{\partial x}$, $p_2 = \frac{\partial z}{\partial x_2}$ $\ldots$ $p_n = \frac{\partial z}{\partial x_n}$ & $P_1, P_2 \ldots P_n$ and R are the functions of $x_1, x_2 \ldots x_n$ & z then the n^{th} order Lagrange A.E become, $\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R}$

Then $u_1 = c_1$, $u_2 = c_2 \dots, u_n = c_n$ be 'n' independent solution of the above A·E

then the G.S is $\phi(u_1, u_2 \dots u_n) = 0$

① Solve $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = xyz$

ans: Here the Lagrange A-E is

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \frac{du}{R}$$

i.e $\frac{dx}{P_1} = \frac{dy}{P_2} = \frac{dz}{P_3} = \frac{du}{R}$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz} \quad \text{———} ①$$

Forgetting the 1st solution we are taking the 1st & 2nd ratio

$\frac{dx}{x} = \frac{dy}{y}$, Then on integration

$$\int\frac{dx}{x} = \int\frac{dy}{y}$$

i.e $\log x = \log y + \log c_1$

$\log x = \log (y \times c_1)$

i.e $x = y c_1 \Rightarrow \frac{x}{y} = c_1$ ——②

Forgetting the 2nd solution we are taking the 2nd & 3rd ratio

$\frac{dy}{y} = \frac{dz}{z}$ Then on integration

$\int\frac{dy}{y} = \int\frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$

$y = z \times C_2 \Rightarrow \frac{y}{z} = C_2$ ——— ③

which is the 2nd independent solution.

For getting 3rd solution we are taking yz, zx, xy as multipliers in ①

$$\frac{yzdx + zxdy + xydz}{xyz + xyz + xyz} = \frac{yzdx + zxdy + xydz}{3xyz} \text{ ——— ④}$$

Now we are taking ④ with the last fraction of ①

$$\frac{yzdx + zxdy + xydz}{3(xyz)} = \frac{du}{xyz}$$

$$yzdx + zxdy + xydz = 3du$$

$$d(xyz) = 3du$$

Then on integration we get

$$xyz = 3u + C_3$$

$$xyz - 3u = C_3 \text{ ——— ⑤}$$

which is the 3rd independent solution.

Then the G.S is $\phi(u_1, u_2, u_3) = 0$

i.e. $\phi\left(\frac{x}{y}, \frac{y}{z}, xyz - 3u\right) = 0$

28/9/16 Monday

Solution of higher order homogeneous PDE with constant coefficient

An equation of the form

$$\boxed{a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = g(x,}$$ ①

where $a_0, a_1, a_2 \ldots a_n$ are constants is called the higher order P.D.E with constant coefficients.

then by replacing D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial y}$

we get.

$$\boxed{\left[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n\right] z = g(x,y)}$$

ie $F(D,D') = g(x,y)$

the solution of equation ① consisting of two part

(i) The Complementary function (C.F) which is the complete solution (C.S) of the equation $F(D,D')z = 0$ It must contains 'n' arbitary function. where 'n' is the order of the D.E. ★①

(ii) The Particular Integral (P.I) which is the particula solution (free from arbitary functions) of, the equation $F(D, D')z = 0$.

The complete Solution (C.S), $C.S = C.F + P.I$ $(D>D')$ ✱

For finding C.F. (where the power of D' must be $> D$,

For finding C.F, we are putting $\underline{D = m \; \& \; D' = 1}$ (i)

in the eqn $[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n]z = 0$

The we get,

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0 \quad \text{———} \ ②$$

$$F(m, 1) = 0.$$

Then ② is called the auxillary equation (A.E) of the equation ①

Let $m_1, m_2, \dots m_n$ are the distinct roots of ②
$\phi_1, \phi_2 \dots \phi_n$ = arbitary function.

Case 1

If $m_1, m_2 \dots m_n$ are distinct then the C.F.

$$C.F = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \dots + \phi_n(y+m_nx)$$

Note:

If $m_1 = \frac{a_1}{b_1}$, $m_2 = \frac{a_2}{b_2}$, $\dots$, $m_n = \frac{a_n}{b_n}$

Then

$$C.F = \phi_1(b_1y + a_1x) + \phi_2(b_2y + a_2x) \dots + a_n(b_ny + a_nx)$$

Case 2

If One root m_0 is repeating. Then the C.F is.

$$CF = \phi_1(y+m_0x) + x \times \phi_2(y+m_0x) + \dots + x^{k-1}\phi_k(y+m_0x)$$

Remark 1

① If the power of D' is greater than (D'>D) the power of D. the A.E can be found by putting D = 1 & D' = m in the equation ①

The C.F can be written by exchanging x & y.

* Remark 2.

If the power of D & D' are the same. Then the A.E can be found by either putting

(i) D = m & D' = 1 $\boxed{\text{OR}}$ (ii) D = 1, & D' = m.

① Solve the P.D.E $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0.$

ans: The Symbolic form (S.F) can be written by replac

D for $\frac{\partial}{\partial x}$ & D' for $\frac{\partial}{\partial y}$

ie $(D^4 - D'^4)z = 0$.

Now we are writing the Auxillary Equation (AE

by putting $D = m$ & $D' = 1$

the AE is, $m^4 - 1^4 = 0$

Then solving for 'm'.

$$(m^2 - 1^2)(m^2 + 1^2)$$

$m = \pm 1$ $\quad m = \pm i$

ie, The roots of $m = \underset{m_1}{\underset{\downarrow}{1}}, \underset{m_2}{\underset{\downarrow}{-1}}, \underset{m_3}{\underset{\downarrow}{i}}, \underset{m_4}{\underset{\downarrow}{-i}}$

The complementary function CF,

$$CF = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \phi_3(y + m_3x) + \phi_4(y + m_4x$$

$$C.F = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix)$$

② Solve $(D^3 - 4D^2D' + 4DD'^2)z = 0.$

ans: SF, $(D^3 - 4D^2D' + 4DD'^2)z = 0$ $\quad D > D'$

A.E, $D = m$, $D' = 1$.

The AE is, $m^3 - 4m^2 1 + 4m\,1^2 = 0$

$$m^3 - 4m^2 + 4m = 0$$

$$m(m^2 - 4m + 4) = 0$$

$$m = 0, \qquad m^2 - 4m + 4 = 0$$

$$m = \frac{4 \pm \sqrt{4^2 - 4 \times 1 \times 4}}{2 \times 1}$$

$$= \frac{4}{2} = 2, 2$$

$$m = \underset{m_1}{\underset{\downarrow}{0}}, \underbrace{2, 2}_{m_2}$$

$$C.F = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + x\,\phi_3(y + m_2 x)$$

$$= \phi_1(y) + \phi_2(y + 2x) + x\,\phi_3(y + 2x)$$

Solve $(4D^2 + 12DD' + 9D'^2)z = 0$.

A.E, $D = m, \quad D' = 1$.

$4m^2 + 12m + 9 = 0$ or $(2m + 3)^2 = 0$

$m = -\frac{3}{2}, -\frac{3}{2}$

$\therefore m_0 = \frac{a}{b} = \frac{-3}{2}$, putting $a = -3$, $b = 2$.

$z = \phi_1(by + ax) + x\,\phi_2(by + ax)$.

C.S, $z = \phi_1(2y - 3x) + x\,\phi_2(2y - 3x)$.

$\boxed{\text{OR}}$. $D = 1, \quad D' = m$

$4 + 12m + 9m^2 = 0 \qquad (3m + 2)^2 = 0$

$m = -\frac{2}{} \quad -\frac{2}{}$

$m_0 = \frac{a}{b} = \frac{-2}{3}$ $a = -2,\ b = 3$

$C.F = \phi_1(bx+ay) + y\phi_2(bx+ay)$

C.S is

$C.S = \phi_1(3x-2y) + y\phi_2(3x-2y)$

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thursday

④ Solve. $(D^2D' - 4DD'^2 + 4D'^3)z = 0$

ans.

Here power of D' is $> D$, $D'^3 > D^2$

By putting
$D = 1,\ D' = m.$

∴ A.E is. $(m - 4m^2 + 4m^3)\cancel{z} = 0.$

$m(\cancel{2}1 - 4m + 4m^2) = 0$

~~m=0~~, ~~$m = \frac{4 \pm \sqrt{(4)^2 - 4\times1\times4}}{2\times1}$~~

~~$= \frac{4 \pm \sqrt{16-16}}{2} = 2\ \&\ 2$~~

$m\ (4m^2 - 4m + 1) = 0$

$m \neq 0,\quad m = \frac{4 \pm \sqrt{4^2 - 4\times4\times1}}{2\times 4}$

$= \frac{4}{8} = \frac{2}{4} = \frac{1}{2}\ \&\ \frac{1}{2}$

$m = \underset{m_1}{0}, \underbrace{\frac{1}{2}, \frac{1}{2}}_{m_2}$

~~Here~~

Here C.F, $z = \phi(y+m_1 x) + \phi_2(y+m_2 x) + x\,\phi_3(y+m_2 x$

$z = \phi_1(y) + \phi_2\left(y + \frac{1}{2}x\right) + x\,\phi_3\left(y + \frac{1}{2}x\right)$

$z = \phi_1(g) + \phi_2(2y + x) + x\,\phi_3(2y + x)$

) Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$, given that $\frac{\partial z}{\partial x} = a \sin y$, &

$\frac{\partial z}{\partial y} = 0$ when $x = 0$.

∴ The given P.E is. $\frac{\partial^2 z}{\partial x^2} \cancel{z} - a^2 z = 0$

ie, ~~By~~ replacing $\frac{\partial}{\partial x} = D$.

C.F ⇒

$D^2 z - a^2 z = 0.$

$(D^2 - a^2) z = 0.$

A.E ⇒ Here in S.F, no D', So, $D = m$.

$m^2 - a^2 = 0.$

$(m+a)(m-a) = 0$

$m = -a, \quad m = +a. \Rightarrow m = \pm a.$

∴ The G.S is given by

C.F ie $z = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$.

$z = \phi_1(y + ax) + \phi_2(y - ax)$ —— ①

Now we are finding the $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ by partially differentiating the ① w.r.to x & y.

$\frac{\partial z}{\partial x} = a\cdot\phi_1'(y+ax) + (-a)\,\phi_2'(y-ax)$

$\frac{\partial z}{\partial x} = a\left[\phi_1'(y+ax) - \phi_2'(y-ax)\right]$ —— ②

$$\frac{\partial z}{\partial y} = [\phi_1'(y+ax) \times 1 + \phi_2'(y-ax) \times 1]$$

$$\frac{\partial z}{\partial y} = \phi_1'(y+ax) + \phi_2'(y-ax) \text{ ——— } ③$$

Using the given conditions, we have

$$a[\phi_1'(y+ax) - \phi_2'(y-ax)] = a \sin y \text{ — } Ⓐ$$

$$\phi_1'(y+ax) + \phi_2'(y-ax) = 0 \text{ — } Ⓑ$$

Solving the above two equations.

Adding Ⓐ & Ⓑ,

$$\phi_1' 2(y+ax) = \sin y. -$$

$$\phi_1'(y+ax) = \frac{1}{2} \sin y.$$

Substracting Ⓐ & Ⓑ. Ⓑ - Ⓐ

$\phi_2'(y-ax) - - \phi_1$

$$2\phi_2'(y-ax) = -\sin y.$$

$$\phi_2'(y \mp ax) = -\frac{1}{2} \sin y.$$

$$\phi_2'(y \mp ax) = -\phi_1'(y+ax).$$

The above condition becomes. $x = 0$,

$$\phi_1'(y) = \frac{1}{2} \sin y.$$

$$\phi_2'(y) = -\frac{1}{2} \sin y.$$

Then on integration the above eqn becomes

$$\phi_1(y) = \frac{1}{2} \int \sin y$$

$$\phi_1(y) = -\frac{1}{2} \cos y$$

$\phi_2(y) = \frac{-1}{2}\int \sin y$

$\phi_2(y) = \frac{1}{2}\cos y$

These gives $\phi_1(y+ax) = \frac{-1}{2}\cos(y+ax)$

$\phi_2(y-ax) = \frac{1}{2}\cos(y-ax)$

Thence from the ①.

$z = -\frac{1}{2}\cos(y+ax) + \frac{1}{2}\cos(y-ax)$

$\therefore z = \frac{1}{2}\left[\cos(y-ax) - \cos(y+ax)\right]$

the particular Integral (P.I)

Here we are considering the P.D.E

$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots \dots a_n D'^n) z = g(x,y)$

ie, $(F(D, D'))z = g(x,y)$

- In O.D.E, the same time we have the form $f(D)y = g(x)$
- Then P.I, ie $y = \frac{1}{F(D)} g(x)$

Then P.I ie,

$$\boxed{z = \frac{1}{F(D,D')} g(x,y)}$$

Here F(D,D') can be treated just like the operator F(D) in case of the O.D.E with constant coefficients.

Theoram

The general method for finding the P.I

If $g(x,y)$ is a function of x & y, then

P.I ie, $\frac{1}{D - mD'} g(x,y) = \int g(x, c-mx)\,dx$.

Then after performing the integration we are

Replacing the 'c' by '$y+mx$' ($c=y+mx$)

Now. as in O.D.E depending upon the form of $g(x,y)$ shorter methods are available for the particular integral. which are discussed

Case 1

When $g(x,y)$ is of the form

$$g(x,y) = e^{ax+by}.$$

$$P.I = \frac{1}{F(D,D')} e^{ax+by} \quad [D=a,\ D'=b]$$

$$= \frac{1}{F(a,b)} e^{ax+by}, \quad f(a,b) \neq 0$$

If $f(a,b) = 0$, then $F(D,D')$ must have a factor of the type $(bD-aD')^r$. In this case the P.I is obtained by using the particular formula i.e P.I

$$P.I = \frac{1}{(bD-aD')^r} e^{ax+by}$$

$$P.I = \frac{x^r}{b^r r!} e^{ax+by}.$$

Note:

If in the case of $f(a,b)=0$ & it cannot be converted to the form $(bD-aD')^r$

Then the P.I can be obtained by partially differentiating the denominator with D or D' and simultaneously multiplying the numerator by

x or y. i.e,

$$P \cdot I = x \cdot \frac{1}{f'(a,b)} e^{ax+by}.$$

where $f'(a,b) = \left[\frac{\partial}{\partial D} f(D, D')\right]_{(a,b)} \neq 0.$

case-2

When $g(x,y)$ is of the form

$$g(x,y) = \cos(ax+by) \text{ or } \sin(ax+by)$$

$$P \cdot I = \frac{1}{F(D^2, DD', D'^2)} \cos(ax+by) \text{ or } \sin(ax+by).$$

$D^2 = -a^2$

$DD' = -ab.$

$D'^2 = -b^2.$

$$P \cdot I = \frac{1}{f(-a^2, -ab, -b^2)} \cos(ax+by) \text{ or } \sin(ax+by)$$

$g(x) = \cos ax$ or $\sin ax$.

$P \cdot I = \frac{1}{F(D)} \cos ax$ or $\sin ax$

$D^2 = -a^2$

$P \cdot I = \frac{1}{F(-a^2)} \cos ax$ or $\sin ax$

$F(-a^2) \neq 0$

If $F(-a^2) = 0$, $= 0$ case of failure

$P \cdot I = x \cdot \frac{1}{F'(-a^2)}$

if $f'(-a^2) \neq 0$

$$f(-a^2, -ab, -b^2) \neq 0$$

If $F(-a^2, -ab, -b^2) = 0$

$$P \cdot I = \frac{1}{(bD - aD')^r} \cos(ax+by) \text{ or } \sin(ax+by).$$

$$= \frac{x^r}{b^r r!} \overset{ax+by}{\cdot} \cos(ax+by) \text{ or } \sin(ax+by).$$

Note: If $f(D^2, DD', D'^2)$ cannot be transformed to the form $(bD - aD')^r$, then we can proceed as the above, case I.

① Solve the following D·E e^{5x+6y}

$$(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x+6y}$$

ans: The problem is already given in the S.F.

A·E is, found by putting $D^3 = D'^3$

$D = m$ & $D' = 1$.

ie, The A·E,

For solving m $m^3 - 6m^2 + 11m - 6 = 0$.

$F(m) = m^3 - 6m^2 + 11m - 6$

$f(1) = 1 - 6 + 11 - 6$ $\quad 1 - 6 + 11 - 6$

$= \underline{\underline{0}}$.

$m = 1$ is a root.

$(m-1)$ is a factor

Then by synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$m^2 - 5m + 6 = 0$

$m = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5+1}{2}$ or $\frac{5-1}{2}$

$= \frac{6}{2}$ or $\frac{4}{2}$

$\underline{3}$ $\quad \underline{2}$

$\underline{\underline{m = 2, 3}}$

$\therefore I \cdot F = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \phi_3(y + m_3x)$

$CF = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$

$$P.I = \frac{1}{F(D,D')} e^{ax+by}.$$

$$= \frac{1}{D^3 - 6D^2 + 11DD'^2 - 6D'^3} e^{5x+6y}.$$

$[D = a = 5, \; D' = b = 6.]$

$$= \frac{1}{5^3 - 6\times5^2\times6 + (11\times5\times6^2) - 6\times6^3}$$

$$= \frac{1}{\cancel{125 - 1350 + 1980 - 1296 -}}$$

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$$= \frac{1}{-91} e^{5x+6y}$$

② Solve $(D - 2D')(D - D')^2 z = e^{x+y}.$ $(a-b)^2$

an. $D^3 - 2D^2D' + DD'^2 - 2D^2D' + 4DD'^2 - 2D'^3$

$SF \Rightarrow (D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{x+y}$

$AE \Rightarrow D^3 = D'^3$ (power)

$\therefore D = m, \; D' = 1.$

$m^3 - 4m^2 + 5m - 2 = 0 \Rightarrow f(m) = m^3 - 4m^2 + 5m - 2 = 0$

$f(1) = 1 - 4 + 5 - 2 = 0$

$\therefore m = 1$ is a root &

$(m-1)$ is a factor

By synthetic division,

$$\begin{array}{r|rrrr} 1 & 1 & -4 & 5 & -2 \\ & 0 & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

$m^2 - 3m + 2 = 0.$

$$m = \frac{3 \pm \sqrt{(-3)^2 - 4\times1\times2}}{2\times1}$$

$$= \frac{3 \pm \sqrt{1}}{2}$$

$$= \frac{3+1}{2} \;\&\; \frac{3-1}{2} = \frac{4}{2} \;\&\; \frac{2}{2}$$

$= 2 \;\&\; 1$

Roots are, $m = 2, 1, 1$ //

$$C.F = \phi_1(y+2x) + \phi_2(y+x) + x\,\phi_3(y+x).$$

$$P.I = \frac{1}{F(D,D')} e^{ax+by}.$$

$$= \frac{1}{(D-2D')(D-D')^2} e^{x+y}$$

$[D = a = 1, \quad D = b = 1]$

$$= \frac{1}{(1-2)\ (1-1)^2} e^{x+y}$$

$$= \frac{1}{0} e^{x+y},$$ the case of failure.

Here converting $F(D,D')$ into the form $(bD-aD')^r$ is not simple. So we are going to the method of differentiation.

~~$P.I = x \cdot \frac{1}{(D-2D') \times 2(D-D')}$~~

$\frac{\partial(uv)}{\partial x} = u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial u}{\partial x}$

$u = D-2D'$

$V = (D-D')^2$

$$P.I = x \cdot \frac{1}{\underbrace{(D-2D')}_{u} \cdot 2\underbrace{(D-D')}_{v} \times 1 + (D-D')^2 \times 1} e^{x+y}$$

$[D = a = 1, \quad D' = b = 1]$

$$= x \cdot \frac{1}{(1-2)\ 2(1-1)1 + (1-1)^2}$$

$$= x \cdot \frac{1}{0} e^{x+y}$$ the case of failure

$$P.I = x^2 . \frac{1}{2[(D-2D')\times 1+(D-D')\times 1]+2(D-D')\times 1} e^{x+y}$$

$$[D=a=1, \quad D'=b=1]$$

$$= x^2 \frac{1}{2[-1+0+0]}$$

$$= \underline{\underline{\frac{-x^2}{2} e^{x+y}}}$$

∴ The G.S,

$$z = C.F + P.I$$

$$z = \phi_1(y+2x) + \underline{\underline{\phi_2(y+x) + x\phi_3(y+x)}} - \frac{x^2}{2} e^{x+y}$$

solve.

③ $$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

Ans: The problem is already not ~~the S.Eq.~~ in the S.F,

∴ The S.F is, $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

$D^2 = D'^2$

A.E is, $D = m, \; D' = 1$

$$m^2 - 4m + 4 = 0$$

$$m = \frac{4 \pm \sqrt{(-4)^2 - 4\times 1\times 4}}{2\times 1}$$

$$= \frac{4 \pm}{2}$$

$$= 2 \; \& \; 2$$

Case 3

Shorter Method for finding P.I when $g(x,y)$ is of the form $\phi(ax+by)$ i.e

$$g(x,y) = \phi(ax+by)$$

When $g(x,y)$ is of the form $\phi(ax+by)$

i.e, $P\cdot I = \frac{1}{F(D,D')}\phi(ax+by).$

$$= \int\cdots\iint \frac{1}{F(a,b)} \phi(v)\, dv^n.$$

$\boxed{v=ax+by}$

$D=a, D'=b.$

$$= \frac{1}{F(a,b)} \int\cdots\iint \phi(v)\, dv^n$$

where 'n' is the degree of the homogeneous function $f(D,D')$ & after performing the integration, i.e $\phi(v)$ n times v is replaced by $ax+by$ i.e $v=ax+by$

9/4/16 **Note.**

If $F(a,b)=0$, then the RHS becomes infinite & the above formula fails. In such cases $F(D,D')$ must be factorised into factors of the for $(bD-aD')^n$. ie our P.I becomes

$$P.I = \frac{1}{F(D,D')}\phi(ax+by)$$

$$= \frac{1}{(bD-aD')^n}\phi(ax+by)$$

$$= \underline{\underline{\frac{x^n}{n!.b^n}\phi(ax+by)}}$$

1) Solve, $(D^2+3DD'+2D'^2)z = 2x+3y$.

ans The problem is already in S.F.

A.E, $D^2 = D'^2$

$D = m, \quad D' = 1.$

$\therefore m^2 + 3m+2 = 0$

$$\therefore m = \frac{-3\pm\sqrt{9-4\times1\times2}}{2\times1}$$

$$= \frac{-3\pm1}{2} = \frac{-3+1}{2} \;\&\; \frac{-3-1}{2}$$

$$= \cancel{2} \;\&\; \cancel{1} \;\; \underline{\underline{-1}} \;\&\; \underline{\underline{-2}}$$

C.F ie

$$z = \underline{\underline{\phi_1(y-x) + \phi_2(y-2x)}}$$

Here, $P.I = \frac{1}{F(D,D')}\phi(ax+by)$

(power=2)

$\frac{1}{F(D,D')}\phi(ax+by)$

$= \frac{1}{F(a,b)}\iint v\,dv\,dv$

$v=2x+3y.$

$= \frac{1}{D^2+3DD'+2D'^2}\phi(2x+3y)$

~~D=a=~~

$[D=a=2, D'=b=3]$

$= \frac{1}{2^2+3\times2\times3+2\times3^2}\iint v\,dv\,dx.$

$v=2x+3y.$

$= \frac{1}{40}\int\frac{v^2}{2}dy.$

$= \frac{1}{80}\int v^2 dv$

$= \frac{1}{80}\times\frac{v^3}{3} = \frac{1}{240}v^3 = \frac{1}{240}\times(2x+3y)^3.$

∴ the complete solution

$z = \phi_1(y-x)+\phi_2(y-2x)+\frac{1}{240}(2x+3y)^3$

Solve $r+s-2t=\sqrt{2x+y}$

ie case-3 $\sqrt{} = \phi(ax+by)$

SF⇒

$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\,\partial y}-2\frac{\partial^2 z}{\partial y^2}=(2x+y)^{1/2}$

$m=1, -2$

AE $D^2+DD'-2D'^2=(2x+y)^{1/2}$

$D=m, D'=1$

$m^2+m-2=0$

$\sqrt{9}$

$m=\frac{-1\pm\sqrt{1-4\times1\times-2}}{2\times1}=\frac{-1\pm3}{2}$

$m = \underline{-1+3}$ & $\underline{-1-3}$ = $\underline{2}$ & $\underline{-4}$ = 1 & -2 ie CF ie, $z = \phi_1(y+x) + \phi_2(y-2x)$

degree = 2

$$\frac{1}{f(D,D')}\phi(ax+by)$$

$$= \frac{1}{f(a,b)} \iint \phi(v)\, dv\, dv$$

$\phi(v) = (2x+y)^{1/2}$

$\Rightarrow v = 2x+y$

$v = 2x+y$

$$P.I = \frac{1}{F(D,D')}\phi(ax+by).$$

$$= \frac{1}{D^2+DD'-2D'^2}(2x+y)^{1/2}$$

$[D = a = 2,\ D' = b = 1]$

$$= \frac{1}{2^2+2\times1-2\times1^2} ~~(2x+y)^{1/2}~~ \iint (v)^{1/2}\, dv\, dv$$

$\frac{1}{2}+1$

$\frac{1+2}{2}$

$$= \frac{1}{4}\int \frac{v^{1/2+1}}{1/2+1}\, dv$$

$\frac{3}{2}+1 = \frac{5}{2}$

$$= \frac{1}{4}\int \frac{v^{3/2}}{3/2}\, dv$$

$$= \frac{2}{12}\int v^{3/2}\, dv = \frac{1}{6}\frac{v^{5/2}}{5/2}$$

$$= \frac{2}{30} v^{5/2}$$

$$= \frac{2}{30}(2x+y)^{5/2}$$

$$= \frac{1}{15}(2x+y)^{5/2}$$

∴ the C.S, $z = C.F + PI$

$$z = \phi_1(y+x) + \phi_2(y-2x) + \frac{1}{15}(2x+8)^{5/2}$$

3) Solve $(D^2 - 2DD' + D'^2)z = \tan(y+x)$

AE, $D = m,\ D' = 1$

$m = 1, 1$

$$m^2 - 2m + 1 = 0$$

$$m = \frac{2 \pm \sqrt{4-4}}{2} = 1, \ 1$$

$C.F = \phi_1(y+x) + x\,\phi_2(y+x)$

$$P.I = \frac{1}{f(D,D')}\,\phi(ax+by)$$

$$= \frac{1}{D^2-2DD'+D'^2}\,\tan(y+x)$$

$$\boxed{\begin{aligned}&\frac{1}{F(D,D')}\,\phi(ax+by)\\ &= \frac{1}{f(a,b)}\iint \phi(v)\,dv\,dv\\ &\phi(v) = \tan(y+x)\end{aligned}}$$

$[D = a = 1,\ D' = b = 1]$

$$= \frac{1}{0}\,\tan(y+x),$$ the case of failure

$(D-D')^2$

$$P.I = \frac{1}{F(D,D'}\,\tan(y+x)$$

$$\frac{x^n}{b^n \times n!}$$

$$= \frac{1}{(D-D')^2}\,\tan(y+x)$$

$(bD-aD')^n$

This $(D-D')^2 = (bD-aD')^n$ $\therefore n = 2$, $b = 1$

$$= \frac{x^2}{2! \times 1^2}\,\tan(y+x)$$

$$\boxed{\frac{x^n}{n!\,b^n}\,\phi(ax+by)}$$

$$= \frac{x^2}{2}\,\tan(y+x)$$

$\therefore$ C.S, i.e $z = CF + P.I.$

$$z = \phi_1(y+x) + x\,\phi_2(y+x) + \frac{x^2}{2}\tan(y+x)$$

6/4/ Module 1 Partial Differential Equation
(Continuum)

Method of finding P.I when $g(x,y) = x^m y^n$

As discussed in the ordinary D.E, take out the lowest degree term from $f(D,D')$ so as to reduce it in the form $[1 \pm F(D,D')]^n$

After that take it to the numerator & get $[1 \pm F(D,D')]^{-n}$ which is then expanded with the help of binomial theorem.

If $g(x,y) = x^m y^n$, then $[1 \pm F(D,D')]^{-n}$ is expanded in powers of $\frac{D'}{D}$ if $m > n$ & expanded in powers of $\frac{D}{D'}$ if $m < n$.

1) Solve $(D^2 - 2DD' - 15D'^2)z = 12xy$

ans: A.E ; $D = m$, $D' = 1$ $(m = -3, 5)$

$m^2 - 2m - 15 = 0$.

$m = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times -15}}{2 \times 1}$

$= \frac{2 \pm \sqrt{4 + 60}}{2}$

$= \frac{2 \pm 8}{2}$

$= \frac{2+8}{2}$ & $\frac{2-8}{2}$

$\therefore m = \underline{\underline{5}}$ & $\underline{\underline{-3}}$

C.F $= \phi_1(y \cancel{+} -3x) + \phi_2(y + 5x)$

$$P.I = \frac{1}{F(D, D')} x^m y^n \qquad [1 \pm F(D,D')]^n$$

$$= \frac{1}{D^2 - 2DD' - 15D'^2} \times 12xy \qquad x^1 y^1 = x^m y^n$$

$m = 1$, $n = 1$

$$= 12 \frac{1}{D^2\left[1 - \frac{2D'}{D} - 15\frac{D'^2}{D^2}\right]} xy$$

$$= \frac{12}{D^2} \frac{1}{\left[1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right]^1} xy$$

$$\boxed{(1-x)^{-1} = 1 + x + x^2 + x^3 \ldots}$$

$$= \frac{12}{D^2} \left[1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{12}{D^2} \left[1 + \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right] xy$$

only power, $m = 1$, $n = 1$ not 2

$$= \frac{12}{D^2} \left[xy + \frac{2D'}{D} xy + \cancel{\frac{15D'^2}{D^2} xy}\right]$$

$$\underline{\underline{0}}$$

$D' =$ w.r.to y

$= \underline{\underline{x}}$

$$\frac{15 D'^2}{D^2} xy = D'^2(xy) = D'(x) = \underline{\underline{0}}$$

$$= \frac{12}{D^2} \left[xy + \frac{2}{D} x + \frac{15}{D^2} \times 0\right]$$

$$= 12 \left[\frac{1}{D^2}(xy) + \frac{2}{D^3}(x)\right]$$

$$\frac{1}{D} xy = \frac{x^2}{2} y$$

$$\frac{1}{D}\left(\frac{x^2}{2}\right)$$

$$= 12 \left[\frac{1}{D} \frac{x^2}{2} y + \frac{2}{D^2} \left(\frac{x^2}{2}\right)\right]$$

$$= 12 \left[\frac{x^3}{6} y + \frac{2}{D} \left(\frac{x^3}{6}\right)\right] = +2$$

$$= 12\left[\frac{x^3}{6}y + 2\frac{x^4}{\cancel{24}_{12}}\right]$$

$$= \frac{12}{6}x^3y + \frac{x^4}{\cancel{12}} \times 12$$

$$= 2x^3y + x^4$$

Friday 3/4/16

2) Find the real function V of x & y, reducing to zero when y = 0. and satisfying.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi\,(x^2+y^2).$$

Ans: Here the real function V satisfying the given equation can be obtained by finding the P.I of the given eq. equation.

The symbolic of the given equation is

$$(D^2 + D'^2)\,V = -4\pi\,(x^2+y^2)$$

$$P.I = \frac{1}{F(D, D')}\,g(x, y) = x^m y^n$$

m < n

$\frac{D}{D'}$

$x^2y^0 + y^2 \cdot x^0$

$$= \frac{1}{D^2+D'^2}\,-4\pi\,(x^2+y^2)$$

$$\left[1 \pm F(D, D')\right]^{1}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$= -4\pi\,\frac{1}{D^2\left[1+\frac{D'^2}{D^2}\right]}\,(x^2+y^2)$$

$r = 2 \quad (x^2, D^2, D'^2)$

$$= \frac{-4\pi}{D^2} \times \left[1+\frac{D'^2}{D^2}\right]^{-1} \times (x^2+y^2)$$

$$= \frac{-4\pi}{D^2}\left[1 - \frac{D'^2}{D^2}\right] \quad (x^2+y^2)$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2) - \frac{D'^2}{D^2}(x^2+y^2)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2) - \frac{1}{D^2}*(2)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2) - \frac{1}{D}\cdot\frac{1}{D}(2)\right]$$

$$= -\frac{4\pi}{D^2}\left[(x^2+y^2) - 2\frac{1}{D}x\right]$$

$$= -\frac{4\pi}{D^2}\left[(x^2+y^2) - 2\frac{x^2}{2}\right]$$

$$= -\frac{4\pi}{D^2}\left[(\cancel{x^2}+y^2 - \cancel{x^2})\right.$$

$$= \left[-\frac{4\pi}{D^2}y^2\right]$$

$$= \left(-4\pi\frac{1}{D}\left(\frac{1}{D}y^2\right)\right)$$

$$= -4\pi\frac{1}{D}y^2x.$$

$$= -4\pi\, y^2\frac{x^2}{2}$$

$$= \underline{\underline{-2\pi\, \cancel{y}\, x^2y^2}}.$$

$\therefore V = -2\pi x^2y^2$

with reduces to zero when $y=0$,

$$V = -2\pi \times x^2 \times 0 = \underline{\underline{0}}$$

$D'($

$D' =$

$D'^2 =$

$D'^2(x$

$\frac{1}{D}$

$\frac{1}{D}$

$\frac{1}{D} =$

Finding The method of Finding P·I when $g(x, y)$ is of the form $e^{ax+by}\cdot V(x,y)$

$$g(x,y) = e^{ax+by}\cdot V(x,y).$$

$$P\cdot I = \frac{1}{F(D,D')}\, e^{ax+by}\cdot V(x,y).$$

ording D·E

$$g(x) = e^{ax}\cdot V$$

$$P\cdot I = \frac{1}{F(D)}\left(e^{ax}\cdot V\right)$$

$$PI = e^{ax}\cdot \frac{1}{F(D+a)} V.$$

$$P\cdot I = e^{ax+by}\cdot \frac{1}{F(D+a,\, D'+b)} V(x,y)$$

Solve $(D^2 + DD' - 6D'^2)z = y \sin x.$

$SF \Rightarrow D^2 + DD' - 6D'^2.$

$D = m,\ D' = 1$ $\qquad m = 2, -3$

$\therefore A\cdot E \Rightarrow m^2 + m - 6 = 0$

$$m = \frac{-1 \pm \sqrt{1^2 - 4\times1\times-6}}{2\times 1}$$

$$= \frac{-1 \pm \sqrt{\cancel{24}1+24}}{2} = \frac{-1+5}{2} \ \&\ \frac{-1-5}{2}$$

$$= \frac{4}{2} \ \&\ -\frac{6}{2} = \underline{\underline{2}} \ \&\ \underline{\underline{-3}}$$

$$C\cdot F,\ z = \underline{\underline{\phi_1(y+2x)}} + \phi_2(y-3x)$$

$$P\cdot I = \frac{1}{F(D,D')}\, g(x,y).$$

$$P\cdot I = \frac{1}{F(D,D')}\, g(x,y)$$

$$= I\cdot P\text{ of } e^{ix}\, xy$$

Sinx · x y

$e^{ix} = \cos x + i\sin x$

$\sin x$ = Imaginary part of e^{ix}

i.e,

$\sin x = I\cdot P$ of e^{ix}

$$P\cdot I = \frac{1}{D^2+DD'-6D'^2}\; I\cdot P \text{ of } e^{ix}\, xy$$

Note:

$e^{ix} = \cos x + i\sin x$

$\cos x$ = Real Part of e^{ix}

i.e $\cos x = R\cdot P$ of e^{ix}

$$= I\,P\text{of}\, e^{ix+0y}\, \frac{1}{(D+i)^2+(D+i)(D'+0)-6(D'+0)^2}\, xy$$

(ix) D changed to $D+i$

D' ,, to $D'+0$ $(0y)$

$$= I\,P\text{of}\, e^{ix+0y}\, \frac{1}{D^2+2Di+i^2+(DD'+0+D'i+0)-6D'^2}\, xy$$

$$= I\,P\text{of}\, e^{ix}\, \frac{1}{D^2+2iD-1+DD'+iD'-6D'^2}\, xy$$

$$= I\cdot P\text{ of } e^{ix}\, \frac{1}{(D^2+DD'-6D'^2)+(2iD+iD')-1}\, y$$

No need to take

$* \; y = y^1x^0$

$m=0,\ n=1$

$m<n$

$\hookrightarrow \frac{D}{D'}$

const

$$= I\cdot P\text{ of } e^{ix}\, \frac{1}{D^2+DD'-6D'^2+i(2D+D')-1}$$

multiplying by taking minus (minus)

$$(1\pm F(D,D'))^n$$

$$= -I\cdot P\text{ of } e^{ix}\, \frac{1}{1-[i(2D+D')+D^2+DD'-6D'^2]}$$

$$= -I\cdot P\text{ of } e^{ix}\, \left[1-(i(2D+D')+D^2+DD'-6D'^2)\right]^{-1}$$

$= -I.P \text{ of } e^{ix}\left[1 + i(2D+D') + D^2 + DD' - 6D'^2\right]y$ $\boxed{(1-x)^{-1} = 1 + x + x^2.}$

$= -I.P \text{ of } e^{ix}\left(y + i(2D'y) + D'(y)\right)$

$= -IP \text{ of } e^{ix}\left[y + i(1)\right]$ $\boxed{e^{ix} = \cos x + i\sin x}$

$= -I.P \text{ of } \left[(\cos x + i\sin x)\ (y + i)\right].$

$= -I.P \text{ of } \left[y\cos x + i\cos x + i\sin x\, y + i^2\sin x\right]$

[illegible] $I.P \text{ of } \left[y\cos x + i\cos x + y\sin x - \sin x\right]$

$= -(y\sin x + \cos x)$

Solve $\left(D^2 + DD' - 2D'^2\right)z = (y-1)e^{x}.$

A.E, $D = m, \quad D' = 1.$

$m^2 + m - 2 = 0$ $\qquad m = -1, 2.$

$m = \dfrac{+1 \pm \sqrt{1^2 - 4\times1\times-2}}{2\times1}$

$= \dfrac{+1 \pm \sqrt{1+8}}{2}$

$= \dfrac{+1+3}{2} \ \& \ \dfrac{+1-3}{2} = \dfrac{4}{2} \ \& \ \dfrac{-2}{2} = 2 \ \& \ -1$

C.F ie $z = \phi_1(y-x) + \phi_2(y+2x).$

$P.I = \dfrac{1}{F(D,D')}\, e^{ax+by}\, .\, V(x,y).$

$= \dfrac{1}{D^2 + DD' - 2D'^2}(y-1)e^{x}.$ $\qquad D \to D+1$ $\quad D' \to D'+0$

$= e^{1x+0y}\dfrac{1}{(D+1)^2 + (D+1)(D'+0) - 2(D'+0)^2}(y-1).$

$$= e^{x} \frac{1}{D^2 + 2D + 1^2 \pm DD' \pm 0 \pm D' + 0 - 2D'^2} (y-1)$$

$$= e^{x} \cdot \frac{1}{D^2 + 1 + 2D - DD' - D' - 2D'^2} (y-1)$$

$$= e^{x} \cdot \frac{1}{[1 + (2D - D' - DD' + D^2 - 2D'^2)]^{1}} (y-1)$$

$$= e^{x} [1 + (2D - D' - DD' + D^2 - 2D'^2)]^{-1} (y-1)$$

$$= e^{x} [1 - \underset{=0}{\cancel{2D}} + D' + \cancel{DD'} + \cancel{D^2} + \cancel{2D'^2}] (y-1)$$

$D'(y-1) = 1 \quad (0)1 = 0$

$$\boxed{(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots}$$

$\boxed{(y-1) \text{ Power of } y = 1}$

$$= e^{x} (1 + D') (y-1).$$

D' w.r.to y

$\rightarrow D'(y-1) = 1$

$$= e^{x} [(y-1) + \underset{1}{D'(y-1)}]$$

$$= e^{x} (y - 1 + 1)$$

$$= e^{x} (y)$$

$$= ye^{x}$$

C.S i.e

$$z = \phi(y - x) + \phi_2 (y + 2x) + ye^{x}.$$

(3) Solve $\dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y} = 3e^{2x} - 2x^2y$

Ans

- $P.I = \dfrac{1}{f(D,D')} f(x,y) = \dfrac{1}{D^3 - 2D^2 D'} (\underset{(P.I_1)}{3e^{2x}} - \underset{(P.I_2)}{2x^2y})$
- $P.I_1 = \dfrac{1}{D^3 - 2D^2 - D'} 3e^{2x+0y}$, $P.I_2 = \dfrac{1}{D^3 - 2D^2 D'} (2x^2y)$ $m=2, n=1$ $m>n \Rightarrow \dfrac{D'}{D}$

A S done in Tutorial of Module

(4) Solve $(D^2 D' - 2DD'^2 + D'^3) z = \frac{1}{x^3}$

$D' = m, \quad D = 1.$

A E Q

$$m^3 - 2m^2 + m = 0 \qquad m = 0, 1, 1$$

$$m(m^2 - 2m + 1) = 0$$

$$m = 0, \quad m^2 - 2m + 1 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 1}}{2 \times 1}$$

$$m = \frac{2}{2} \& \frac{2}{2} \Rightarrow \underline{\underline{m = 1, 1}}.$$

C.F is $z = \phi_1(y) + \phi_2(y+x) + x\phi_3(y+x).$

$$P.I = \frac{1}{F(D, D')} g(x, y).$$

$$= \frac{1}{D^2 D' - 2DD'^2 + D'^3} \left(\frac{1}{x^3}\right)$$

$$= \frac{1}{\underline{D'}(D^2 - 2DD' + D'^2)} \left(\frac{1}{x^3}\right) \qquad \frac{1}{D'}\left(\frac{1}{x^3}\right) = \underline{\underline{\frac{1}{x^3} \times y}}$$

$$= \frac{1}{D^2 - 2DD' + D'^2} \left(\frac{y}{x^3}\right).$$

$$= \frac{1}{(D - D')^2} \left(\frac{y}{x^3}\right)$$

$$\frac{1}{D - mD'} g(x, y) = \int g(x, c - mx)$$

remaining c

$= \frac{1}{\quad} \quad 1 \quad \times \left(y\right)$

$$= \frac{1}{(D-D')} \int \left(\frac{c-x}{x^3} \right) dx.$$

$$\boxed{\begin{matrix} D = m \\ y = c - x \end{matrix}}$$

5) Solv

$$= \frac{1}{(D-D')} \int \left(\frac{c}{x^3} - \frac{1}{x^2} \right) dx.$$

$$* \int c \frac{1}{x^3}\, dx$$

$$c \int x^{-3} dx = \frac{x^{-2}}{-2}$$

$$= \frac{-1}{2x^2}$$

ins.

$$= \frac{1}{(D-D')} \quad c \cdot \log x \left(\frac{-c}{2x^2} - -\frac{1}{x} \right)$$

$$* \int \frac{1}{x^2} dx =$$

$$\int x^{-2} dx = \frac{x^{-1}}{-1}$$

$$\frac{-2+1}{-2+1} \frac{-1}{1} = -\frac{1}{x}$$

$$= \frac{1}{D-D'} \left[\frac{-c}{2x^2} + \frac{1}{x} \right]$$

$$c = y + mx$$

$$\Rightarrow c = y + x$$

$$= \frac{1}{D-D1} \left[\frac{-(y+x)}{2x^2} + \frac{1}{x} \right]$$

$(D - mD')$ case

y change to $\Rightarrow \boxed{\begin{matrix} m = 1, \\ y = c' - \cancel{m} x \end{matrix}}$

$$= \int \frac{-(c' - x + x)}{2x^2} + \frac{1}{x} dx.$$

$$-c \int \frac{1}{2x^2} dx = -\frac{c}{2} \int x^{-2}$$

$$= \frac{-c}{2} \times \frac{x^{-1}}{-1}$$

$$= \frac{c}{2} \times \frac{1}{x}$$

$$x^{-1+1}$$

$$= \int \frac{-c'}{2x^2} + \frac{1}{x} dx$$

$$= \frac{c'}{2x} + \log x$$

Remann

$$c = y + mx.$$

$$c' = y + x.$$

$$= \frac{y+x}{2x} + \log x = \frac{y}{2x} + \frac{1}{2} + \log x$$

$\therefore$ The C.S is $z = \phi_1(y) + \phi_2(y+x) + \phi_3(y+x) + \frac{y}{2x} + \frac{1}{2} + \log x.$

(5) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (2x^2 + xy - y^2)\sin xy - \cos xy$

ans) A.F $\Rightarrow$ $(D^2 - DD' - 2D'^2)z = 0$

A.E $\Rightarrow$ $m^2 - m - 2 = 0$ $(m = -1, 2)$

$$m = \frac{1 \pm \sqrt{1 - 4 \times 1 \times -2}}{2 \times 1}$$

$$= \frac{1 \pm 3}{2}$$

$$= \frac{1+3}{2} \ \& \ \frac{1-3}{2}$$

$$= \underline{\underline{2}} \ \& \ \underline{\underline{-1}}$$

ere C.F ie $z = \phi_1(y - x) + \phi_2(y + 2x)$.

$$P.I = \frac{1}{F(D, D')} g(x, y).$$

$$= \frac{1}{D^2 - DD' - 2D'^2} (2x^2 + xy - y^2)\sin xy - \cos xy$$

general case

$$\frac{1}{D - mD'} g(x, y) = \int g(x, c - mx)$$

$c \to y + mx$

$y = c - m$

$$= \frac{1}{(D - 2D')(D + D')} \left([(2x - y)(x + y)] \sin xy - \cos xy \right)$$

$D - mD'$

$m = (-1)$

$c = y + mx$

$c = y - x \Rightarrow y = \underline{c + x}$

$D - mD'$

$D + D' \Rightarrow \underline{\underline{m = -1}}$

$y = c - mx$

$y = \underline{c + x}$

$$= \frac{1}{D - 2D'} \int \left[\underbrace{2x - (c + x)}_{x - c} (x + c + x) \sin x(c + x) - \cos x(c + x) \right]$$

$$= \frac{1}{D - 2D'} \int \left[(x - c)\underbrace{(2x + c). \sin(cx + x^2)}\, - \cos(cx + x^2) \right]$$

$$= \frac{1}{D-2D'} \int \left[(x-c)\left(-\frac{d}{dx}\cos(cx+x^2)\right)\right] dx - \int \cos(cx+x^2)\,dx.$$

$\int u\,v\,dv$

$$= \frac{1}{D-2D'} \left\{(x-c)(+\cos(cx+x^2))\right\} + \boxed{\int \cos(cx+x^2)\,dx} - \cancel{\int \cos(cx+x^2)\,dx}.$$

$\boxed{\int u\,dv = uv_1 - u'v_2 + u''}$

$$= \frac{1}{D-2D'} (c-x)\cos(cx+x^2)$$

$\boxed{\begin{array}{l} c = y+mx \\ c = y+x \\ y = c-x \\ (m=-1) \end{array}}$

$$= \frac{1}{D-2D'} (y-x-x)\cos\left(\overset{(y-x)x.}{\cancel{cx}} + x^2\right)$$

$$= \frac{1}{D-2D'} (y-2x)\cos(xy - \cancel{x^2} + \cancel{x^2})$$

$y = y+mx.$

$y = y+2x$

$y = c$

$y = c - mx$

$$= \frac{1}{\underset{D-mD'}{D-2D'}} (y-2x)\cos xy$$

$\underline{y = c-2x}$

$$= \int (c'-2x-2x)\cos x\cancel{y}(c'-2x)\,dx.$$

$$= \int \underbrace{(c'-4x)\cos(c'x-2x^2)}_{\text{derivative of sin}}\,dx.$$

$$= \int \frac{d}{dx}\sin(c'x-2x^2)\,dx.$$

$\boxed{c' = y+2x}$

$$= \sin(c'x-2x^2)$$

$$= \sin((y+2x)x-2x^2)$$

throughc

$$p = \frac{\partial z}{\partial x} = z_x, \quad q = \frac{\partial z}{\partial y} = z_y$$

$$r = \frac{\partial^2 z}{\partial x^2} = z_{xx}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = z_{xy}.$$

$$t = \frac{\partial^2 z}{\partial y^2} = z_{yy}.$$

Order & Degree of a Partial D·E (P·D·E)

The order of a P·D·E, is the order of the higest partial derivative present in that equation.

The degree of a P·D·E, is the degree (power) of the highest order partial derivative present in the equation, when the equation has been made rational & integral as far as the partial derivatives are concerned. For e.g:

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$; order = 1, degree = 1

$\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + \sin x$; order = 2, degree = 1

$\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} y = \left(\frac{\partial z}{\partial y}\right) z$; order = 1, degree = 2

Number of variable.

The number of variables is the number of independent variables in the P.D.E. For eg:

① $u_t = u_x$. u → dependent variable ; t, x – independent variables

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$ $> u = f(t, x)$ = 2 variables

② $u_t = \alpha^2 [u_{xx} + u_{yy} + u_{zz}]$

$$\frac{\partial u}{\partial t} = \alpha^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right]$$

u → dependent
t, x, y, z → indepe

No of variables = 4 variables

$u = f(t, x, y, z)$

* Linear and non-linear P.D.E

The P.D.E, are classified as being linear & non-linear. A P.D.E, is said to be linear if the dependent variable & its partial derivative are of degree 1. i.e, they appear in linear fashion. and are not multiplied together raised to powers etc. More generally, the general second order P.D.E, in two variable x & y takes the form

$$\boxed{Au_{xx} + Bu_{xy} + Cu_{yy} + Du_{x} + Eu_{y} + Fu = G} \text{ — (A)}$$

where A, B, C, D, E, F and G are given functions x & y. otherwise, the P.DE is said to be non-linear

For eg:

① $u_t = c^2 u_{xx}$. → linear

② $u_{xx} + u_{yy} + u = \sin x$. → linear

③ $u u_x + y u_y + u = 1$. → non-linear.

④ $u_{xx} + u_{yy} + u^2 = 0$. → non-linear.

- Homogeneous & Non-Homogeneous P.D.E

- In the general P.D.E Ⓐ is homogeneous if $G = 0$ for all x & y in the domain of the equation. otherwise it is said to be non-homogeneous.

Now we are arranging the eqn Ⓐ are

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x\,\partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

- For homogeneous case $G = 0$ the equation becomes →

$$\left(A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x\,\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F\right)u = 0$$

$$Lu = 0 \text{ — Ⓑ}$$

where, 'L' is the differential operator i.e

$$L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial x\,\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F$$

5/16 Friday The Principle of Super Position

If $u_1, u_2 \ldots u_n$ be an infinite solutions of the equation Ⓑ then

$$\sum_{n=1}^{\infty} C_n u_n = C_1 u_1 + C_2 u_2 + \cdots\cdots + C_n u_n$$

converges and is differentiable term by term through out some domain of the independent variable, then ~~u=~~

$u = \sum_{n=1}^{\infty} C_n u_n$ is also a solution of equation Ⓑ

P.D.E(s) with constant and variable coefficients.

If the coefficient A, B, C, D, E & F in eqn. Ⓐ a all constants then the equation is said to be a P.D.E with constant coefficients, otherw it is a P.D.E with variable coefficienl

Eg:

1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \sin x$ → P.D.E with constant coefficients (constant=1)

2) $x \frac{du}{dx} + y \frac{du}{dx} = 1.$ → P.D.E with variable coefficients (constants= x

Formation of P.D.E

Now we are going to form a P.D.E

Eliminating arbitary constants
Eliminating arbitary function.

ormation by eliminating arbitary constants.

Suppose we have an equation

$f(x, y, z, a, b) = 0$ —— ⓒ

where, 'a', and 'b' are arbitary constants.

Let us consider z as a dependent variable and 'x' and 'y' are independent variable (ie, Two independent variable).

Now we are eliminating the two arbitary constant in ~~eq~~ 'a, b' in the equation ⓒ by substituting the suitable P.Derivative with respect to x & with respect to y. Then we get after elimination $\phi(x, y, z, p, q) = 0$. —— ⓓ which is a P.D.E of first order

ie, equation ⓒ & ⓓ show that if the number arbitary constants to be eliminated is equal to the number of indypendent variable, then we get a first order P.D.E.

otherwise, if the no of arbitary constants ~~to~~ be eliminated is more than, the no f independent variable, we get a PDE of higher order. However, this highe

Note:

If the no of arbitary constants is mor than two, then the obtained second orde partial derivative to eliminate arbitary constant.

① Find the P.D.E by eliminating the arbitary constants 'a' and 'b' from the following.

(a) $z = (x-a)^2 + (y-b)^2$

(b) $z = ax + by + a^2 + b^2$

ans;

(a) x, y - independent variable, a, b - arbitary constant

$$z = (x-a)^2 + (y-b)^2 \quad \text{——} ①$$

Differentiating the ① partially with respect to x

$$\frac{\partial z}{\partial x} = 2(x-a) + 0 \qquad \boxed{\frac{\partial z}{\partial x} = p}$$

$$p = 2(x-a) \Rightarrow x - a = \frac{p}{2} \quad \text{——} ②$$

Differentiating the ① partially w.r. to 'y'.

$$\frac{\partial z}{\partial y} = 0 + 2(y-b)$$

$$q = 2(y-b) \Rightarrow y - b = \frac{q}{2} \quad \text{——} ③$$

Substituting the eqn(s) ② & ③ in ①

$$\therefore z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$$

$$z = \frac{p^2}{4} + \frac{q^2}{4}$$

$$z = \frac{p^2+q^2}{4}$$

$\Rightarrow 4z = p^2+q^2$ which is the required P.D.E with first order

(ii) $z = ax + by + a^2 + b^2$ —— ①

Differentiating the eqn ① partially w.r.to x.

$$\frac{\partial z}{\partial x} = a \longrightarrow$$

$\rightarrow p = a$ —— ②

Differentiating the eqn ① partially w.r.to 'y'

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \text{ —— ③}$$

Substituting the eqn(s) ② & ③ in ①.

$\therefore z = px + \underline{\underline{qy +}} p^2 + q^2$, which is the required P.D.E with first order. PDE

② Find the D.E for all spheres of fixed radius having their centres in the $\underline{XY \text{plane}}$

ans: The equation of the sphere with centre at $(h, k, 0)$ in the xy-plane & radius 'r' is given by.

$$(x-h)^2 + (y-k)^2 + z^2 = r^2 \longrightarrow ①$$

Differentiating the ① partially w.r. to x.

$$2(x-h)+0+2z\frac{\partial z}{\partial x}=0$$

$$2\left[(x-h)+z\frac{\partial z}{\partial x}\right]=0$$

$$(x-h)+zp=0$$

$$x-h=-zp \quad \text{②}$$

Differentiating ① partiall w. r. to y:

$$0+2(y-k)+2z\frac{\partial z}{\partial y}=0$$

$$2\left[(y-k)+z\frac{\partial z}{\partial y}\right]=0$$

$$y-k+zq=0$$

$$y-k=-zq \quad \text{③}$$

Substituting ② & ③ in ①

$$\Rightarrow \quad (-zp)^2+(-zq)^2+z^2=r^2$$

$$z^2p^2+z^2q^2+z^2=r^2$$

$$\underline{z^2(p^2+q^2+1)}=r^2$$, which is the required First order PDE

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Tuesday

Formation of PDE by eliminating the arbitary fuction.

1 Q. Form the PDE by eliminating the arbitary function.

① $z=f(x^2-y^2)$

? $z = f(x^2-y^2)$ —①

Differentiating the eqn ① partially w. r. to x.

$$\frac{\partial z}{\partial x} = f'(x^2-y^2) \times 2x$$

$$\Rightarrow P = 2x \times f'(x^2-y^2) \text{ —② }$$

Differentiating the eqn ① partially w. r to y.

$$\frac{\partial z}{\partial y} = f'(x^2-y^2) \times -2y.$$

$$q = -2y \times f'(x^2-y^2) \text{ —③ }$$

Dividing

②/③ $\Rightarrow$ $\frac{P}{q} = \frac{-x}{y}$ $\Rightarrow py = -qx$

$\Rightarrow \underline{py + qx = 0}$.

which is a first order P.D.E.

$z = f\left(\frac{x.y}{z}\right)$

$z = f\left(\frac{xy}{z}\right)$ —①

Differentiating eqn ① partially with respect to x

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right) \times \overset{\text{quotient rule}}{\frac{\partial}{\partial x}\left(\frac{xy}{z}\right)}$$

$$p = f'\left(\frac{xy}{z}\right)\left[\frac{z \times \frac{\partial}{\partial x}(xy) - xy \times \frac{\partial z}{\partial x}}{z^2}\right] \qquad \boxed{\frac{\partial z}{\partial x} = p}$$

$$p = f'\left(\frac{xy}{z}\right)\left[\frac{z \times (y \times 1) - xy \times p}{z^2}\right]$$

$$p = y \times f'\left(\frac{xy}{z}\right) \times \left(\frac{z - xp}{z^2}\right) \text{ —② }$$

Similarly Differentiating eqn ① partially w. r. to y.

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right) \frac{\partial}{\partial y}\left(\frac{xy}{z}\right)$$

$$q = f'\left(\frac{xy}{z}\right)\left[\frac{z \times \frac{\partial}{\partial y}(xy) - (xy) \times \frac{\partial z}{\partial y}}{z^2}\right]$$

$$q = f'\left(\frac{xy}{z}\right)\left[\frac{z \times (x \times 1) - xy \times q}{z^2}\right] \qquad \boxed{\frac{\partial z}{\partial y} = q}$$

$$q = x\cancel{q} \times f'\left(\frac{xy}{z}\right)\left[\frac{z - yq}{z^2}\right] \text{ --- ③}$$

$$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{y(z-xp)}{x(z-yq)} \Rightarrow \frac{p}{q} = \frac{yz - xyp}{xz - xyq}$$

$$\cancel{p(x(z-yq))} = q\,(y(z-xp))$$

$$\cancel{px(pz-pqy)} = qy\,\cancel{(qz-pzx)}$$

$$p(xz - xyq) = q(yz - xyp)$$

$$\cancel{p^2xz - p^2xy} = pxz - xypq - qyz + xypq = 0$$

$$(px - qy)z = 0.$$

$$\Rightarrow \therefore \underline{px - qy = 0}$$

⑧ $z = f(x+iy) + g(x-iy), \quad i = \sqrt{-1}$

ans: $z = f(x+iy) + g(x-iy)$ —— ①

Differentiating the ① partially w.r.to x.

$$\frac{\partial z}{\partial x} = f'(x+iy) \times 1 + g'(x-iy) \times 1$$

$$p = f'(x+iy) + g'(x-iy) \text{ —— ②}$$

Differentiating the ② partially w.r.to y

$$\frac{\partial z}{\partial y} = f'(x+iy) \times i + g'(x-iy) \times -i.$$

$$q = i\left[f'(x+iy) - g'(x-iy)\right] \text{ —— ③}$$

Differentiating the ② again partially w.r.to x.

$$\boxed{\frac{\partial^2 z}{\partial x^2} = r}$$

$$\boxed{\frac{\partial^2 z}{\partial x \partial y} = s}$$

$$\boxed{\frac{\partial^2 z}{\partial y^2} = t}$$

ie, $r = \frac{\partial^2 z}{\partial x^2} = f''(x+iy) \times 1 + g''(x-iy) \times 1$

$r = f''(x+iy) + g''(x-iy)$ —— ④

Differentiating the ③ $\overset{\text{partially}}{\text{with}}$ respect to y.

$t = \frac{\partial^2 z}{\partial y^2} = i[f''(x+iy) \times i - g''(x-iy) \times -i]$

$t = i^2 [f''(x+iy) + g''(x-iy)]$ $\boxed{i^2 = -1}$

$t = -[f''(x+iy) + g''(x-iy)]$ —— ⑤

∴ from the eqn ④ and ⑤, we have

$r = -t$, ie $r + t = 0$,

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$, which is a second orde P.D.E

Form the P.D.E from the equation

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, by eliminating the arbitary constants.

(e no. of independent variable to be eliminated = arbitary constants)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$, —— ①

Differentiating ① partially w. r. to x.

$\frac{2x}{a^2} + 0 + \frac{1}{c^2}\left(2z \frac{\partial z}{\partial x}\right) = 0 \Rightarrow \frac{2x}{a^2} + \frac{1}{c^2}\left(2z \cdot \frac{\partial z}{\partial x}\right) = 0$ ②

Diff. the ② partially w. r. to y

$0 + \frac{2}{c^2}\left(\frac{\partial}{\partial y} z \cdot \frac{\partial z}{\partial x}\right) = 0$

In this problem the numbe of arbitary constants to be eliminated = 3, which is greater than the number of independent variable whic is equal to 2 (x, y)

(product rule) of differentiation

ie; $\frac{2}{c^2}\left[z \cdot \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}\right] = 0$

ie,
$$\frac{2}{c^2}\left[z\frac{\partial^2 z}{\partial y\,\partial x} + \frac{\partial z}{\partial x} \cdot \frac{\partial z}{\partial y}\right] = 0$$

Since, $\frac{2}{c^2} \neq 0$ (c is arbitary constant $c \neq 0$)

$$\Rightarrow \underline{z\frac{\partial^2 z}{\partial y\,\partial x} + \frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 0.}$$

which is a P.D.E of second order.

3) Form the P.D.E by eliminating the arbitary functions in, $z = yf(x) + x \cdot g(y)$

ans: $z = y\,f(x) + x \cdot g(y)$ —— ①

Differentiating the ① partially w.r.to x.

$$\frac{\partial z}{\partial x} = y \cdot f'(x) \times 1 + g(y) \times 1$$

$$p = y\,f'(x) + g(y) \text{ —— ②}$$

Differentiating the ① partially w.r.to y!

ie, $\frac{\partial z}{\partial y} = f(x) \times 1 + x \times g'(y) \times 1$

$$q = f(x) + x\,g'(y) \text{ —— ③}$$

Differentiating the ② partially w.r.to y.

$$\frac{\partial^2 z}{\partial x\,\partial y} = s = f'(x) \times 1 + g'(y) \text{ —— ④}$$

Eliminating the $f'(x)$ and $g'(x)$ from in eqn ④ with the help of the eqn(s) ② & ③

$$f'(x) = \frac{p - g(y)}{y}, \qquad g'(y) = \frac{q - f(x)}{x}$$

$$s = \left[\frac{p - g(y)}{y}\right] + \left[\frac{q - f(x)}{x}\right]$$

$$s = \frac{x(p - g(y)) + y(q - f(x))}{xy}$$

$$s = \frac{xp - x \times g(y) + qy - y \times f(x)}{xy}$$

from eqn ①

$$z = y \times f(x) + x \times g(y).$$

$$xy \cdot s = \overset{px}{\cancel{xp}} + qy - x\,g(y) - y\,f(x)$$

$$xys = px + qy - z$$, which is a P.D.E of second order.

Formation of PDE by eliminating the arbitary function 'ϕ' from the given equation $\phi(u, v) = 0$, where u & v are the funcition of x, y and z.

* * *

Here u, v are the two functions containing the '3' independent variables x, y & z. and 'ϕ' is an arbitary function.

By eliminating the arbitary function 'ϕ' from the relation $\phi(u, v) = 0$ we get a first order P.D.E in p, q and ... is the form '$Pp + Qq = R$' where P, Q, R are the functions of the 3 independent variables x, y & z wher ~~Po~~

.yz.
zx,
xy.

$$\boxed{\begin{aligned} P &= \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y} \\ Q &= \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z} \\ R &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\times\frac{\partial v}{\partial x} \end{aligned}}$$

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= ①

③ Q. Find the D.E of all planes which are ~~not~~ at a constant distance from the origin.

ans. The eqn of all planes which are at a constant distance 'k' units from origin takes the form, $lx + my + nz = k$ — ①

where $l^2 + m^2 + n^2 = 1^2$

Differentiating ① with respect to x & y

$l \cdot 1 + n\frac{\partial z}{\partial x} = 0 \Rightarrow l \cdot 1 + np = 0$ —②

n-partially

$m + n\frac{\partial z}{\partial y} = 0 \Rightarrow m + nq = 0$ —③

eliminating l, m, and n using ①, ② & ③

① becomes $(-np)x + (-nq)y + nz = k\sqrt{l^2+m^2+n^2}$

Q.
①
an

$n(-px - qy + z) = k\sqrt{(-np)^2 + (-nq)^2 + n^2}$

$n(-px - qy + z) = kn\sqrt{p^2 + q^2 + 1}$

$z = px + qy + k\sqrt{p^2 + q^2 + 1}$, which is partial D.E of first order.

Note: In all of the above eg's, the number of arbitary constants is equal to the number of dependent variables. Hence the partial D.E(s) are of first order.

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Continues

Note.

① If the given relation b/w x, y & z contains two arbitary functions & then in general there elimination will give a second order p.D.E

② If the arbitary function are more than 2, then we need the higher order p.Derivatives for elimination.

② The relation $\phi(u, v) = 0$ involving the arbitary function ϕ can be written in another form like $u = \phi(v)$ or $v = \phi(u)$

after elimination of the arbitary function ϕ we get a first order ~~in~~ p & q is of the form $Pp + Qq = R$

whare
$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$\therefore p = 2y \times 2z - 2z \times (-x)$$

$$P = \underline{\underline{4yz + 2xz}}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}$$

$$= 2z \times (-y) - 2x \cdot 2z$$

$$= -2yz - 4xz$$

$$Q = \underline{\underline{-(2yz + 4xz)}}$$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}$$

$$= 2x \times (-x) - 2y \times (-y)$$

$$= -2x^2 + 2y^2$$

$$R = \underline{\underline{2(y^2 - x^2)}}$$

The required P.D.E is of the form $p = \frac{\partial^2 x}{\partial z^2}$

$$(4yz + 2xz)p - (2yz + 4xz)q = 2(y^2 - x^2)$$

$$(2yz + xz)p - (yz + 2xz)q = y^2 - x^2$$

whic is of first orde

⑫ Form a P.D.E by eliminating the arbitary function ϕ from $x.y.z = \phi(x+y+z)$

ans: The given relation is of the form,

$$u = \phi(v)$$

where $u = x.y.z$, $v = x+y+z.$

after elimination the arbitary function ϕ from the given relation we get a first order p.D.E is of the form $Pp + Qq = R.$

$$P = \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial z} - \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial y}$$

$$= xz \times 1 - xy \times 1.$$

$$P = xz - xy \Rightarrow P = \underline{\underline{x(z-y)}}$$

$$Q = \frac{\partial u}{\partial z} \cdot \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial z}.$$

$$= xy \times 1 - yz \times 1.$$

$$Q = xy - yz \Rightarrow Q = \underline{\underline{y(x-z)}}$$

$$R = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x}.$$

$$= yz \times 1 - xz \times 1.$$

$$R = yz - xz \rightarrow R = \underline{\underline{z(y-x)}}$$

After elimination,
∴ the required p.D.E is of the form,

$$x(z-y)\,p + y(x-z)\,q = z(y-x)$$

which is a P.D.E of first order.

Solution of first order P.D.E's

Before going to solve a P.D.E we have to discuss about some kinds of solutions of P.DE

① Complete Integral.

We have seen that the elimination of the arbitary constant from the relation ~~f(x,y,z)~~

$$f(x,y,z,a,b) = 0. \text{———} Ⓐ$$

Gives the following P.D.E

$$\phi(x,y,z,p,q) = 0 \text{———} Ⓑ$$

If the equation Ⓐ is derived from the equation Ⓑ by any method. Then the eqn Ⓐ which contains the arbitary constant that are equal to the number of independent variable is called The complete ~~the~~ integral of the equation Ⓑ

② Particular Integral

A particular integral of the eqn Ⓑ is obtained b giving particular values to the constants a b in the complete integral.

③ General Integral

The elimination of the arbitary function 'φ' from the relation $\phi(u,v)=0$, where u and v are the function of the 3 independent variables x, y, & z gives a linear P.D.E of the form, $P_p + Q_q = R$, If we say that $\phi(u,v)=0$ is the solution of the P.D.E

$P_p + Q_q = R$, then it is know as the general integral.

④ Singular Integral

The envelope of the surface of the equation $f(x,y,z,a,b)=0$ —Ⓐ is obtained by eliminating the arbitary constants a & b from the eqn ~~Res~~ $f=0$, $\frac{\partial f}{\partial a}=0$ & $\frac{\partial f}{\partial b}=0$.

The above eqn gives a relation b/w $x, y,$ & z which is called the singular integral.

Note:

Let the complete integral be $f(x,y,z,a,b)=0$ and let us assume that one of the arbitary Constant be a function of other say $b=\phi(a)$. then our eqn becomes $f(x,y,z,a,\phi(a))=0$

The general integral is obtained by eliminating the arbitary constant 'a' from $f(x,y,z,a,\phi(a))=0$ & $\frac{\partial f}{\partial a}=0$. This eqn gives the curve of intersection of two consecutive surfaces given by the relation $f(x,y,z,a,\phi(a))=0$ This curve is known as the envelope of the family

Lagrange's Method.

consider a first order P.D.E $Pp+Qq=R$ which is linear in p and q were $P, Q, \& R$ are junction of x, y, z. The procedure for the large method is used for finding u & v as the solution o.f the above P.D.E

Theorem.

The general solution of the linear P.D.E $Pp+Qq=R$ is $\phi(u,v)=0$, where ϕ is an arbitary funct and u, v are the function of 3 independent variable x, y & z ie; $u(x,y,z) - c$ & $u(x,y,c)$ are the two independent solution of ~~Lea~~ Lagranges auxillary equation

$$\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}}$$

Working Rule

To obtain the solutions of PDE, $Pp+Qq=R$ from the Lagranges auxillary equation. $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$. Find two independent solution of these auxillary equation as $u(x,y,z)=c$ and $v(x,y,z)=c_2$ and the general solution of P.DE $Pp+Qq=R$, is given by one of the following forms $\phi(u,v)=0$ (or)

$u=\phi(v)$, $v=\phi(u)$.

In order to obtain the two independent solution we use the following methods.

(i) Method of grouping (Rule 1)

Suppose that one of the variable is either absent or cancel out from any pair of the fractions of $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, and then a solution can be obtained by using usual methods. To Same procedure is repeated with another pair of fraction of the above equation for II nd independent solutions.

(ii) Methods of multipliers (Rule 2)

If l, m, n are the three multiplier then by a were none principle of algebra each fraction of eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ will be equal to

$$\frac{ldx + mdy + ndz}{l + m + n}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$. Then $ldx + mdy + ndz = 0$. which can be integrated to give $u(x,y,z) = c$. The method may be repeated to get another independent solution $v(x,y,z) = c_2$, The multipliers may be functions of x, y, z)/constants. Some times, it is required to find only one solution using multipliers.

* Note-1

If one solution of the eqn $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ is determined by rule-1 & suppose that we can't determine the 2nd solution by rule 1 or rule 2 then first solution known to us is used to find the another independent solutions.

* Note - 2

If one solution is found that $g(x,y) = c_1$ then the constant c_1 should be used in the place of $g(x,y)$ throughout the process of calculating the 2nd solution and finally when 2nd solution is found involving c_1 then c_1 should again be replaced by $g(x,y)$

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Monday

① Solve $y^2zp + x^2zq = xy^2$. T

The given eqn is ~~of~~ a Lagrange's D.E of the form $Pp + Qq = R$

and the Lagrange ~~A~~.E is $\boxed{\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}}$

$P = y^2z, \quad Q = x^2z, \quad R = xy^2$

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2} \quad (A.E) \longrightarrow ①$$

taking the first two ratio, we get

$$\frac{dx}{y^2\cancel{z}} = \frac{dy}{x^2\cancel{z}}$$

$$\frac{dx}{y^2} = \frac{dy}{x^2}$$

$$x^2 dx = y^2 dy$$

Then on integration we get

$$\int x^2 dx = \int y^2 dy$$

$$\frac{x^3}{3} = \frac{y^3}{3} + c$$

$$\frac{x^3 - y^3}{3} = c \Rightarrow \therefore x^3 - y^3 = 3c.$$

$$c_1 = 3c \qquad \therefore \underline{x^3 - y^3 = c_1}$$

which is our first independent solution of the form $u(x, y, z) = c_1$

For getting the second independent solution we have to take the first & third ratio

i.e, $\dfrac{dx}{y^2 z} = \dfrac{dz}{x y^2}$

$$\frac{dx}{z} = \frac{dz}{x} \Rightarrow x\,dx = z\,dz.$$

Then on integration we get

$$\int x\,dx = \int z\,dz.$$

$$\frac{x^2}{2} = \frac{z^2}{2} + c$$

$$\frac{x^2 - z^2}{2} = c \Rightarrow x^2 - z^2 = 2c.$$

$$2c = c_2 \Rightarrow \underline{x^2 - z^2 = c_2}.$$

which is our second independent solution of the form $v(x, y, z) = c_2$

Then the general solution $\phi(u, V) = 0$.

ie, $\phi(x^3 - y^3, x^2 - z^2) = 0$

② solve

$$p \tan x + q \tan y = \tan z$$

ans: The given eqn is a Lagrange D.E is of the form

$$P_p + Q_q = R.$$

Then the Lagrange A.E are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ie, $$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad \text{—①}$$

For getting the first indepent solution We are taking the first 2 Ratio

$$\frac{dx}{\tan x} = \frac{dy}{\tan y}$$

Then on integration, $\int \frac{dx}{\tan x} = \int \frac{dy}{\tan y}$

ie, $\int \cot x \, dx = \int \cot y \, dy$

ie, $\log \sin x = \log \sin y + \log c_1$

ie, $\log \sin x = \log(c_1 \sin y)$

$\sin x = c_1 \sin y$

$\int \cot x \, dx =$

$\int \frac{\cos x}{\sin x} dx$

$\int \frac{du}{u}$ $\quad u = \sin x$

$du = \cos x$

$= \log(u) + c$

$\frac{\sin x}{\sin y} = c_1$ which is our first independent solution is of the form $u(x, y, z) = c_1$

forgetting the 2nd independent variable, we ~~gx~~ are taking 2nd & 3rd ratio

$$\frac{dy}{\tan y} = \frac{dz}{\tan z}$$

Then on integration $\int \frac{dy}{\tan y} = \int \frac{dz}{\tan z}$

$$\int \cot y\, dy = \int \cot z\, dz$$

$\log a + \log b = \log a \cdot b$

$$\log \sin y = \log \sin z + \log c_2$$

$$\log \sin y = \log (c_2 \sin z)$$

$$\sin y = c_2 \sin z$$

$\frac{\sin y}{\sin z} = c_2$, which is our 2nd independent solution is of the form $V(x, y, z) = c_2$.

Then our general solution $\phi(u, v) = 0$.

i.e, $\phi\left(\frac{\sin x}{\sin y}, \frac{\sin y}{\sin z}\right) = 0$

Hw

③ Solve $xp + yq = 3z$.

Hw

④ Solve $x^2p + y^2q = z^2$

⑤ Solve. $p - q = \log(x+y)$

ans: The given eqn is a lagrang D.E is of the form

$$Pp + Qq = R$$

The Lagrange A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)} \quad \text{—①}$$

for getting the 1st independent solution we are grouping the first & second Ratio

i.e $\frac{dx}{1} = \frac{dy}{-1} \Rightarrow dx = -dy$

Then on integration we get,

$$\int dx = \int dy$$

$$x = -y + c_1$$

$$x+y = c_1 \quad \text{—②}$$ ~~which~~ is first eqn $u(x,y,z)=$

For getting the second independent solution we are grouping the first & third ratio

ie, $\frac{dx}{1} = \frac{dz}{\log(x+y)}$

from ② $\Rightarrow$ $dx = \frac{dz}{\log c_1}$

$$\log c_1 \, dx = dz$$

Then on integration, we get

$$\int \log c_1 \, dx = \int dz$$

$$x \log c_1 = z + c_2$$

ie $x \log(x+y) - z = c_2$ ——— ⑧

which is our 2nd independent solution of the form $V(x, y, z) = c_2$

∴ The general solution is of the form $\phi(u, v) = 0$

ie $\phi(x+y, \underline{x \log(x+y) - z}) = 0$.

⑥ Solve $x^2(y-z)p + y^2(z-x)q = z^2(x-y)$

The Given eqn is a Lagrange P.E is of the form

$Pp + Qq = R$

ie A.E $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

ie $$\frac{dx}{x^2(y-z)} = \frac{dy}{y^2(z-x)} = \frac{dz}{z^2(x-y)} \quad ①$$

By method of multipliers

$$\boxed{\begin{array}{c} l, m, n \\ \frac{ldx + mdy + ndz}{lP + mQ + nR} \end{array}} \boxed{\frac{1}{x}, \frac{1}{y}, \frac{1}{z}}$$

~~Taking~~ For getting the $\overset{two}{2}$ independent solution we can't apply the method of grouping, so here we are choosing $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are Multipliers. Then we get

$$\frac{\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z}}{}$$

denominator

$$x(y-z)+y(z-x)+z(x-y)$$

$$\underline{xy}-\underline{xz}+\underline{yz}-\underline{xy}+\underline{xz}-yz=0$$

ie,

$$\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$$

Considering this with any fraction in ①

ie, $\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$

Then on integration

$$\int\frac{dx}{x}+\int\frac{dy}{y}+\int\frac{dz}{z}=0$$

$$\log x+\log y+\log z=\log c_1$$

$$\log(xyz)=\log c_1$$

$\underline{\underline{xyz=c_1}}$ This is our one independe
solution of the form
$u(x,y,z)=c_1$

ans:

③ $x_p+y_q=3z$

The given equation is a Lagrange D.E
of the form $Pp+Qq=R$

Then the Lagrange A.E is,

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$P=x,\ Q=y,$ ~~z~~ $R=3z$

$$\therefore \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

For getting the first independent solution we are taking the first 2 ratio

$$\frac{dx}{x} = \frac{dy}{y}$$

then on integration, $\int \frac{dx}{x} = \int \frac{dy}{y}$

ie, $\log x = \log y + \log c_1.$

$\log x = \log(c_1 \times y)$

$\underline{x = c_1 y}$

$\Rightarrow \underline{\frac{x}{y} = c_1}$ which is our first independent solution is of the form $u(x,y,z) = c_1$.

For getting the 2nd independent variable we are taking 2nd & 3rd ratio

$$\frac{dy}{y} = \frac{dz}{3z}$$

Then on integration, $\int \frac{dy}{y} = \int \frac{dz}{3z}$.

$\log y = \frac{1}{3} \log z + \log c_2$

$\log y = \frac{1}{3} \log (c_2 \times z)$

$y = \frac{1}{3} \times (c_2 \times z) \Rightarrow \underline{\frac{3y}{z} = c_2}$

which is our 2nd independent solution is of the form $v(x,y,z) = c_2$

Then the general solution $\phi(u, v) = 0$

i.e, $\phi\left(\frac{x}{y}, \frac{3y}{z}\right) = 0$

④ $x^2 p + y^2 q = z^2$

an; The given eqn is Lagrage D.E is of the form $Pp + Qq = R$.

Then the Lagrange A.E is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}, \quad P = x^2, \quad Q = y^2, \quad R = z^2$$

$$\therefore \frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}.$$

Taking the first two ratio

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$u = x^2$
$du = 2x\,dx$
$\frac{du}{2x}$

Then on integration, $\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$

i.e, $\int x^{-2}\,dx = \int y^{-2}\,dy$

$$\frac{x^{-2+1}}{-2+1} = \frac{y^{-2+1}}{-2+1} + C_1$$

$$-x^{-1} = -y^{-1}$$

$$-\frac{1}{x} = -\frac{1}{y} + C_1$$

$-\frac{1}{x}+\frac{1}{y}=c_1 \Rightarrow \frac{1}{y}-\frac{1}{x}=c_1$

which is our first independent solution of the form $u(x,y,z)=c_1$

For getting the 2nd Independent solution Taking 2nd & 3rd Ratios

$$\frac{dy}{y^2}=\frac{dz}{z^2}$$

Then on integration $\int\frac{dy}{y^2}=\int\frac{dz}{z^2}$

$$\int y^{-2}dy=\int z^{-2}dz$$

$$\frac{y^{-2+1}}{-2+1}=\frac{z^{-2+1}}{-2+1}+c_2$$

$$-\frac{1}{y}=-\frac{1}{z}+c_2$$

$$\therefore -\frac{1}{y}+\frac{1}{z}=c_2 \Rightarrow \frac{1}{z}-\frac{1}{y}=c_2$$

which is our second ind independent solution of the form $v(x,y,z)=c_2$

Then the general solution $\phi(u,v)=0$

$$\text{i.e., } \phi\left(\left(\frac{1}{y}-\frac{1}{x}\right),\left(\frac{1}{z}-\frac{1}{y}\right)\right)=0$$

⑥ Qu. continue.

For getting the 2nd independent solution we are choosing $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ are multiplier then each fraction in ① will be equal to

$$\frac{\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2}}{(y-z)+(z-x)+(x-y)} = \frac{\frac{dx}{x^2}+\frac{dy}{y2}+\frac{dz}{z^2}}{0}$$

taking the above fraction with any fract.. of ① we get,

$$\frac{dx}{x^2}+\frac{dy}{y^2}+\frac{dz}{z^2} = 0$$

Then on integration we get,

$$\int\frac{dx}{x^2}+\int\frac{dy}{y2}+\int\frac{dz}{z^2} = 0$$

$$-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = C_2$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = -C_2 \qquad \boxed{-c_2 = C_2}$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = C_2$$

which is the 2nd independent solution of the form $v(x,y,z) = C_2$

Then the general solution is

$$\phi\left(x,y,z,\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 0$$

⑦ Solve $(y^2+z^2)p - xyq + xz = 0$

ans: which is of the form $Pp + Qq = R$.

$$(y^2+z^2)p - xyq = -(xz) \qquad \boxed{\begin{aligned}P &= y^2+z^2,\\ Q &= -(xy)\\ R &= -(xz)\end{aligned}}$$

Then the Lagrange A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e, $\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz}$

For getting the first independent solution we are taking the second & third fraction

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{-\cancel{x}z}$$

i.e, $\frac{dy}{y} = \frac{dz}{z}$

Then on integration we get

$$\int\frac{dy}{y} = \int\frac{dz}{z}, \quad \log y = \log z + \log c_1$$

$$\therefore \log y = \log(c_1 \cdot z)$$

$$\therefore y = c_1 \times z$$

$$\therefore \underline{\underline{\frac{y}{z} = c_1}} \text{ — ②}$$

which is the first independent solution of the form $u(x,y,z) = c_1$.

- For getting the 2^{nd} independent solution, we are choosing $\underline{x, y, z}$ are multipliers Then each fraction in ① will be equal to

$$\frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) + y(-xy) + z(-xz)}$$

$$= \underline{x\,dx + y\,dy + z\,dz}$$

i.e $x\,dx + y\,dy + z\,dz = 0$

Then on integration $\int x\,dx + \int y\,dy + \int z\,dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$$

$$\frac{x^2+y^2+z^2}{2} = C_2 \rightarrow x^2+y^2+z^2 = 2C_2$$

$$x^2+y^2+z^2 = C_2. \qquad \boxed{2C_2 = C_2}$$

which is our IInd independent solution of the form $V(x,y,z) = C_2$

G.S is $\phi(u,v) = 0$

$$\phi\left(\frac{y}{z},\ x^2+y^2+z^2\right) = 0.$$

⑧ Solve $xy\,dx + y^2 dy = zxy - 2x^2.$

which is a Lagrange D.E of the form $Pp + Qq = R$.

& the Lagrage A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e, $$\frac{dx}{xy} = \frac{dy}{y^2} = \frac{dz}{zxy - 2x^2} \quad \text{—①}$$

For getting the first independent solution we are choosing the first two fraction we get

$$\frac{dx}{xy} = \frac{dy}{y^2}$$

$$\frac{dx}{x} = \frac{dy}{y}$$

Then on integration we get $\int\frac{dx}{x} = \int\frac{dy}{y}$

$$\log x = \log y + \log c_1$$

$$\log x = \log(y \times c_1)$$

$$\therefore x = y \times c_1 \Rightarrow x = c_1 y$$

$$\therefore \frac{x}{y} = c_1 \quad \text{——} ②$$

which is first independent solution of the form $ub(x, y, z) = c_1$.

For getting the 2nd independ solution we are using the first independent solution, $\underline{x = c_1 y}$

ie For that we ~~are~~ taking the 2nd & 3rd fraction

$$\frac{dy}{y^2} = \frac{dz}{zxy - 2x^2}$$

$$\frac{dy}{y^2} = \frac{dz}{zc_1y^2 - 2c_1^2y^2} \quad \text{by } ②$$

$$\frac{dy}{y^2} = \frac{dz}{y^2(zc_1 - 2c_1^2)}$$

$$dy = \frac{dz}{zc_1 - 2c_1^2}$$

$$dy = \frac{dz}{c_1(z - 2c_1)}$$

$$c_1\,dy = \frac{dz}{z - 2c_1}$$

Then on integration we get $\int c_1\,dy = \int\frac{dz}{z - 2c_1}$

$$c_1 y = \frac{\cancel{c_1}}{-\cancel{2c_1}} \log(z - 2c_1) + \log c_2$$

$$\boxed{x = c_1 y}$$

$$x = \log (z - 2c_1) + \log c_2$$

$$\boxed{c_1 = \frac{x}{y}}$$

$$x = \log \left(z - 2\frac{x}{y}\right) + \cancel{c_2} \log c_2 .$$

$$x = \log\left(\frac{yz - 2x}{y}\right) + \log c_2$$

$$\boxed{\begin{array}{l} e^x = \frac{yz - 2x}{y} \\ ye^x = (yz - 2x) \\ \frac{ye^x}{yz - 2x} = c_2 \end{array}}$$

$$\frac{ye^x}{yz - 2x} = c_2$$

which is our II ~~nd~~ independent solution $V(x, y, z) = c_2$

i.e our G.S $\phi(u, v) = 0$

$$\phi\left(\frac{x}{y}, \frac{ye^x}{yz - 2x}\right) = 0$$

ⓠ Solve $p - 2q = 3x^2 \sin(y + 2x)$

ans The give eqn is a Lagrange's D.E is of the form $Pp + Qq = R$.

Then the Lagrange A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

i.e $$\frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)} \quad ①$$

for getting the 1st independent solution We are taking the first two fraction

$$\frac{dx}{1} = \frac{dy}{-2}$$

i.e $2\,dx = -dy$

then on integration, we get $\int 2\, dx = \int -dy$

$$2x = -y + c_1$$

$$\underline{2x + y = c_1} \quad \text{---} ②$$

which is our first independent solution of the form $u(x,y,z) = c_1$.

for getting the 2nd independent solution we have to make use of 1st independent solution, for that we are taking the 1st and 3rd fraction

by — ②
$\boxed{y + 2x = c_1}$

$$\frac{dx}{1} = \frac{dz}{3x^2 \sin(y+2x)}$$

$$dx = \frac{dz}{3x^2 \sin c_1}$$

$$3x^2 \sin c_1 \; dx = dz.$$

then on integration we get; $\int 3x^2 \sin c_1\, dx = \int dz$

$$3 \sin c_1 \frac{x^3}{3} = z + c_2$$

$$x^3 \sin c_1 = z + c_2$$

$$x^3 \sin c_1 - z = c_2$$

$$\underline{x^3 \sin(y+2x) - z = c_2} \quad \text{---} ③$$

which our 2nd solution of the form

$v(x,y,z) = c_2$,

Then the G.S, $\phi(u,v) = 0$ ie $\phi(2x+y,\ x^3\sin(y+2x) - z) = 0$

are taking the third & last fraction

$$\frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x^2(x^2+y^2+z^2)}$$

$$\frac{dz}{z} = \frac{2(x\,dx + y\,dy + z\,dz)}{x^2+y^2+z^2}$$

$$\frac{dz}{z} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

Then on integration, we get,

$$\int \frac{dz}{z} = \int \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

$u = x^2+y^2+z^2$
$du = 2x\,dx + 2y\,dy + 2zd$

$\int \frac{du}{u} = \log(u)$

$$\log z = \log(x^2+y^2+z^2) + \log c_2$$

$$\therefore \log z = \log((x^2+y^2+z^2) \times c_2)$$

$$\therefore \frac{z}{x^2+y^2+z^2} = c_2 \quad \text{——} ③$$

which is our 2nd independent solution. then the G.S is of the form $\phi(u, v) = 0$

$$\text{ie, } \therefore \phi\left(\frac{y}{z}, \frac{z}{x^2+y^2+z^2}\right) = 0$$

Solve $(y+z)p + (z+x)q = x+y$

This given eqn is a first order Lagrange P.E of the form $Pp + Qq = R$

Then the Lagrange A.E is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\text{ie } \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \quad \text{——} ①$$

For getting two solutions the method of grouping & multipliers are difficult in some way. So, we are setting some other fractions to ①

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{(y+z)-(z+x)} = \frac{dy-dz}{(z+x)-(x+y)} = \frac{dx+dy+dz}{y+z+z+x+x+y}$$

$$\Rightarrow \underset{①}{\frac{dx}{y+z}} = \underset{②}{\frac{dy}{z+x}} = \underset{③}{\frac{dz}{x+y}} = \underset{④}{\frac{dx-dy}{y-x}} = \underset{⑤}{\frac{dy-dz}{z-y}} = \underset{⑥}{\frac{dx+dy+dz}{2(x+y+z)}}$$

For getting the first independent solution we are taking in 4th & 5th ratio

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$$

$$\frac{dx-dy}{-1(x-y)} = \frac{dy-dz}{-1(y-z)}$$

$$\frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$, then on integration

$$\int \frac{dx-dy}{x-y} = \int \frac{dy-dz}{y-z}$$

$u = x-y$
$du = dx-dy$
$\int \frac{du}{u} = \log|u|$

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log(x-y) = \log((y-z) \times c_1)$$

$$\therefore x-y = y-z \times c_1 \Rightarrow \frac{x-y}{y-z} = c_1 \text{ —— } ②$$

which is the 1st independent solution of the for u(x,y,z getting the 2nd independent solution we are taking the fourth and sixth ration (4th & 6th)

$$\frac{dx-dy}{y-x} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\frac{dx-dy}{-(x-y)} = \frac{dx+dy+dz}{2(x+y+z)}$$

$u = x+y+z$

$du = dx+dy+dz$

$\int\frac{dy}{u} = \log u$

$$-2\times\left(\frac{dx-dy}{(x-y)}\right) = \frac{dx+dy+dz}{x+y+z.}$$

$u = x-y$

$du = dx-dy$

$$2\ \log(x-y) = \log(x+y+z)+\log C_2$$

$$\therefore \log(x-y)^{-2} = \log((x+y+z)\times C_2)$$

$$(x-y)^{-2} = ((x+y+z)\times C_2)$$

$$\frac{1}{(x-y)^2(x+y+z)} = C_2. \quad ③$$

Then the G.S is of the form $\phi(u,v) = 0$

i.e, $\phi\left(\frac{x-y}{y-z}, \frac{1}{(x-y)^2(x+y+z)}\right) = 0$

Generalised form of Lagrange's Method.

Let the nth order Lagrange equation is of the form $P_1p_1 + P_2p_2 \ldots\ldots + P_np_n = R$ when $p_1 = \frac{\partial z}{\partial x}$, $p_2 = \frac{\partial z}{\partial x_2}$ $\ldots$ $p_n = \frac{\partial z}{\partial x_n}$ & $P_1, P_2 \ldots P_n$ and R are the functions o $x_1, x_2 \ldots x_n$ & z then the nth order Lagrange A.E become, $\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R}$

Then $u_1 = c_1$, $u_2 = c_2$ ...., $u_n = c_n$ be 'n' independent solution of the above A·E

then the G·S is $\phi(u_1, u_2 \dots u_n) = 0$

① Solve $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = xyz$

ans: Here the Lagrange A·E is

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \frac{dx_3}{P_3} = \frac{du}{R}$$

i.e $\frac{dx}{P_1} = \frac{dy}{P_2} = \frac{dz}{P_3} = \frac{du}{R}$

$$\frac{dx}{\underset{(1)}{x}} = \frac{dy}{\underset{(2)}{y}} = \frac{dz}{\underset{(3)}{z}} = \frac{du}{\underset{(4)}{xyz}} \quad \text{——} ①$$

Forgetting the 1st solution we are taking the 1st & 2nd ratio

$\frac{dx}{x} = \frac{dy}{y}$, Then on integration

$$\int\frac{dx}{x} = \int\frac{dy}{y}$$

i.e $\log x = \log y + \log c_1$

$\log x = \log(y \times c_1)$

i.e $x = y c_1 \Rightarrow \frac{x}{y} = c_1$ —— ②

Forgetting the 2nd solution we are taking the 2nd & 3rd ratio

$\frac{dy}{y} = \frac{dz}{z}$ Then on integration

$\int\frac{dy}{y} = \int\frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$

$y = z \times c_2 \Rightarrow \frac{y}{z} = c_2$ ——— ③

~~Whoshe~~ is the 2nd independent solution.

for getting 3rd solution we are taking yz, zx, xy as multipliers in ①

$$\frac{yz\,dx + zx\,dy + xy\,dz}{xyz + xyz + xyz} = \frac{yz\,dx + zx\,dy + xy\,dz}{3xyz} \text{ ——— } ④$$

Now we are taking ④ with ~~the~~ the last fraction of ①

$$\frac{yz\,dx + zx\,dy + xy\,dz}{3(xyz)} = \frac{du}{xyz}$$

$yz\,dx + zx\,dy + xy\,dz = 3du$

$d(xyz)$ ~~+ d~~ $= 3du$

then on integration we get

$xyz = 3u + c_3$

$xyz - 3u = c_3$ ——— ⑤

Which is the 3rd independent solution.

Then the G.S is $\phi(u_1, u_2, u_3) = 0$

ie $\phi\left(\frac{x}{y}, \frac{y}{z}, xyz - 3u\right) = 0$

28/9/16 Monday Solution of higher order homogeneous PDE with constant coefficient

An equation of the form

$$\boxed{a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \cdots + a_n \frac{\partial^n z}{\partial y^n} = g(x,} \quad ①$$

where $a_0, a_1, a_2 \cdots a_n$ are constants is called the higher order P.D.E with constant coefficients.

then by replacing D for $\frac{\partial}{\partial x}$ and D' for $\frac{\partial}{\partial y}$

we get

$$\boxed{\left[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n\right] z = g(x,y)}$$

ie $F(D, D') = g(x,y)$

the solution of equation ① consisting of two part

(i) The complementary function (C.F) which is the complete solution (C.S) of the equation $F(D,D')z = 0$ It must contains 'n' arbitary function. where 'n' is the order of the D.E.

(ii) The particular Integral (P.I) which is the particular solution (free from arbitary functions) of, the equation $F(D, D')z = 0$.

The complete Solution (C.S), $C.S = C.F + P.I$ $(D > D')$, *

For finding C.F. (where the power of D' must be $> D$,

For finding C.F, we are putting $D = m$ & $D' = 1$ (i)

in the eqn $\left[a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \cdots + a_n D'^n\right] z = 0$

The we get,

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n = 0 \quad \text{——} ②$$

$$F(m, 1) = 0.$$

Then ② is called the auxillary equation (A·E) of the equation ①

Let $m_1, m_2, \dots m_n$ are the distinct roots of ②
$\phi_1, \phi_2 \dots \phi_n$ = arbitary function.

Case 1

If $m_1, m_2 \dots m_n$ are distinct then the C.F.

$$C.F = \phi_1(y+m_1x)+\phi_2(y+m_2x)+\dots+\phi_n(y+m_nx)$$

Note:

If $m_1 = \frac{a_1}{b_1}$, $m_2 = \frac{a_2}{b_2}$, $\dots$, $m_n = \frac{a_n}{b_n}$
Then

$$C.F = \phi_1(b_1y+a_1x)+\phi_2(b_2y+a_2x)\dots+a_n(b_ny+a_nx)$$

Case 2

If One root m_0 is repeating. Then the C.F is,

$$CF = \phi_1(y+m_0x)+x\times\phi_2(y+m_0x)+\dots+x^{k-1}\phi_k(y+m_0x)$$

Remark 1

① If the power of D' is greater than $(D'>D)$ the power of D. the A·E can be found by putting $D = 1$ & $D' = m$ in the equation ①

The C.F can be written by exchanging x & y.

*** Remark 2.**

If the power of D & D' are the same. Then the A·E can be found by either putting

(i) $D = m$ & $D' = 1$ $\boxed{OR}$ (ii) $D = 1$, & $D' = m$.

① Solve the P.D.E $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0.$

ans: The Symbolic form (S.F) can be written by replac

D for $\frac{\partial}{\partial x}$ & D' for $\frac{\partial}{\partial y}$

ie $(D^4 - D'^4)z = 0.$

Now we are writing the Auxillary Equation (AE

by putting $D = m$ & $D' = 1$

The AE is, $m^4 - 1^4 = 0$

Then solving for 'm'.

$$(m^2 - 1^2)(m^2 + 1^2)$$

$m = \pm 1$ $\quad m = \pm i$

ie, The roots of $m = \underset{m_1}{1}, \underset{m_2}{-1}, \underset{m_3}{i}, \underset{m_4}{-i}$

The complementary function CF,

$$CF = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x) + \phi_4(y + m_4 x)$$

$$\underline{\underline{C.F = \phi_1(y + x) + \phi_2(y - x) + \phi_3(y + ix) + \phi_4(y - ix)}}$$

② Solve $(D^3 - 4D^2 D' + 4DD'^2)z = 0.$

ans: SF, $(D^3 - 4D^2 D' + 4DD'^2)z = 0$ $\qquad D > D'$

A.E, $D = m$, $D' = 1.$

The AE is, $m^3 - 4m^2 \cdot 1 + 4m\,1^2 = 0$

$$m^3 - 4m^2 + 4m = 0$$

$$m(m^2 - 4m + 4) = 0$$

$$m = 0, \qquad m^2 - 4m + 4 = 0$$

$$m = \frac{4 \pm \sqrt{4^2 - 4 \times 1 \times 4}}{2 \times 1}$$

$$= \frac{4}{2} = 2, 2$$

$$m = \underset{\substack{\downarrow \\ m_1}}{0}, \quad \underbrace{2, \quad 2}_{m_2}$$

$$C.F = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + x\,\phi_3(y + m_2 x)$$

$$= \phi_1(y) + \phi_2(y + 2x) + x\,\phi_3(y + 2x)$$

Solve $(4D^2 + 12DD' + 9D'^2)z = 0.$

A.E, $D = m, \quad D' = 1.$

$4m^2 + 12m + 9 = 0$ or $(2m + 3)^2 = 0$

$m = -\frac{3}{2}, -\frac{3}{2}$

$\therefore m_0 = \frac{a}{b} = \frac{-3}{2}$, putting $a = -3, b = 2.$

$z = \phi_1(by + ax) + x\,\phi_2(by + ax).$

C.S, $z = \phi_1(2y - 3x) + x\,\phi_2(2y - 3x)$

OR. $D = 1, \quad D' = m$

$4 + 12m + 9m^2 = 0 \qquad (3m + 2)^2 = 0$

$m = -\frac{2}{\;} \quad -\frac{2}{\;}$

$m_0 = \frac{a}{b} = \frac{-2}{3}$ $\qquad a = -2,\ b = 3$

$C.F = \phi_1(bx+ay) + y\phi_2(bx+ay)$

C.S is

$$C.S = \phi_1(3x-2y) + y\phi_2(3x-2y)$$

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④ Solve, $(D^2D' - 4DD'^2 + 4D'^3)z = 0$

ans.

Here power of D' is $> D$, $\qquad D'^3 > D^2$

By putting $D = 1,\ D' = m.$

$\therefore$ A.E is $(m - 4m^2 + 4m^3)\cancel{z} = 0.$

$$m(\cancel{2}1 - 4m + 4m^2) = 0$$

$$\cancel{m=0},\quad \cancel{m = \frac{4 \pm \sqrt{(4)^2 - 4\times1\times4}}{2\times1} = \frac{4 \pm \sqrt{16-16}}{2} = 2\ \&\ 2}$$

$$m(4m^2 - 4m + 1) = 0$$

$$m \neq 0,\quad m = \frac{4 \pm \sqrt{4^2 - 4\times4\times1}}{2\times4}$$

$$= \frac{4}{8} = \frac{2}{4} = \frac{1}{2}\ \&\ \frac{1}{2}$$

$$m = \underset{\overset{\downarrow}{m_1}}{0},\ \underbrace{\frac{1}{2},\ \frac{1}{2}}_{m_2}$$

~~Here~~

Here C.F, $z = \phi_1(y+m_1x) + \phi_2(y+m_2x) + x\,\phi_3(y+m_2x$

$z = \phi_1(y) + \phi_2(y + \frac{1}{2}x) + x\,\phi_3(y + \frac{1}{2}x)$

$z = \phi_1(g) + \phi_2(2y+x) + x\,\phi_3(2y+x)$

) Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$, giventhat $\frac{\partial z}{\partial x} = a \sin y$, &

$\frac{\partial z}{\partial y} = 0$ when $x = 0$.

∴ the given D.E is. $\frac{\partial^2 z}{\partial x^2}$ ~~z~~ $- a^2 z = 0$

ie, ~~By~~ replacing $\frac{\partial}{\partial x} = D$.

C.F ⇒

$D^2 z - a^2 z = 0.$

$(D^2 - a^2) z = 0.$

A.E ⇒ Here in S.F, no D', So, $D = m$.

$m^2 - a^2 = 0.$

$(m+a)(m-a) = 0$

$m = -a$, $m = +a$. $\Rightarrow m = \pm a$.

∴ The G.S is given by

C.F ie $z = \phi_1(y+m_1x) + \phi_2(y+m_2x)$.

$z = \phi_1(y+ax) + \phi_2(y-ax)$ —— ①

Now we are finding the $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ by partially differentiating the ① w.r.to x & y.

$\frac{\partial z}{\partial x} = a\cdot\phi_1'(y+ax) + (-a)\,\phi_2'(y-ax)$

$\frac{\partial z}{\partial x} = a\left[\phi_1'(y+ax) - \phi_2'(y-ax)\right]$ —— ②

$$\frac{\partial z}{\partial y} = [\phi_1'(y+ax) \times 1 + \phi_2'(y \mp ax) \times 1]$$

$$\frac{\partial z}{\partial y} = \phi_1'(y+ax) + \phi_2'(y-ax) \text{ —— } ⑧$$

Using the given conditions, we have

$$a[\phi_1'(y+ax) \pm \phi_2'(y-ax)] = a \sin y \text{ —— } Ⓐ$$

$$\phi_1'(y+ax) + \phi_2'(y-ax) = 0 \text{ —— } Ⓑ$$

Solving the above two equations.

Adding Ⓐ & Ⓑ,

$$\phi_1' \, 2(y+ax) = \sin y.$$

$$\phi_1'(y+ax) = \frac{1}{2} \sin y.$$

Substracting Ⓐ & Ⓑ. Ⓑ − Ⓐ $\phi_2'(y-ax) -- \phi_1$

$$2\phi_2'(y-ax) = -\sin y.$$

$$\phi_2'(y \mp ax) = -\frac{1}{2} \sin y.$$

$$\phi_2'(y \mp ax) = -\phi_1'(y+ax).$$

The above condition becomes. $x = 0$,

$$\phi_1'(y) = \frac{1}{2} \sin y.$$

$$\phi_2'(y) = -\frac{1}{2} \sin y.$$

Then on integration the above eqn becomes

$$\phi_1(y) = \frac{1}{2} \int \sin y$$

$$\phi_1(y) = -\frac{1}{2} \cos y$$

$\phi_2(y) = \frac{-1}{2}\int \sin y$

$\phi_2(y) = \frac{1}{2}\cos y$

These gives $\phi_1(y+ax) = \frac{-1}{2}\cos(y+ax)$

$\phi_2(y-ax) = \frac{1}{2}\cos(y-ax)$

Hence from the ①.

$$z = -\frac{1}{2}\cos(y+ax) + \frac{1}{2}\cos(y-ax)$$

$$\therefore\ z = \frac{1}{2}\left[\cos(y-ax) - \cos(y+ax)\right]$$

the particular Integral (P·I)

Here we are considering the P·D·E

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \dots \dots a_n D'^n) z = g(x,y)$$

ie, $(F(D, D'))z = g(x,y)$

- In O·D·E, the same time we have the form $f(D)\,y = g(x)$
- Then P·I, ie $y = \frac{1}{F(D)} g(x)$

Then P·I ie,

$$\boxed{z = \frac{1}{F(D,D')} g(x,y)}$$

Here $F(D,D')$ can be treated just like the operator $F(D)$ in case of the O·D·E with constant coefficients.

Theoram

The general method for finding the P·I

If $g(x,y)$ is a function of x & y, then

P·I ie, $\frac{1}{D-mD'} g(x,y) = \int g(x, c-mx)\,dx$.

Then after performing the integration we are

Replacing the 'c' by 'y+mx' ($c = y+mx$)

Now. as in O.D.E depending upon the form of $g(x, y)$ shorter methods are available for the particular integral. which are discussed

Case 1

When $g(x,y)$ is of the form

$$g(x, y) = e^{ax+by}.$$

$$P.I = \frac{1}{F(D,D')} e^{ax+by} \qquad [D = a, D' = b]$$

$$= \frac{1}{F(a,b)} e^{ax+by}, \quad f(a,b) \neq 0$$

If $F(a,b) = 0$, then $F(D,D')$ must have a factor of the type $(bD - aD')^r$. In this case the P.I is obtained by using the particular formula i.e P.I

$$P.I = \frac{1}{(bD-aD')^r} e^{ax+by}$$

$$P.I = \frac{x^r}{b^r r!} e^{ax+by}.$$

Note:

If in the case of $f(a,b) = 0$ & it cannot be converted to the form $(bD - aD')^r$

Then the P.I can be obtained by partially differentiating the denominator with D or D' and simultaneously multiplying the numerator by [illegible].

x or y. ie,

$$P.I = x \cdot \frac{1}{f'(a,b)} e^{ax+by}.$$

where $f'(a,b) = \left[\frac{\partial}{\partial D} f(D,D')\right]_{(a,b)} \neq 0.$

Case 2

When $g(x,y)$ is of the form

$$g(x,y) = \cos(ax+by) \quad or \quad \sin(ax+by)$$

$$P.I = \frac{1}{F(D^2, DD', D'^2)} \cos(ax+by) \text{ or } \sin(ax+by).$$

$$D^2 = -a^2$$

$$DD' = -ab.$$

$$D'^2 = -b^2.$$

$$P.I = \frac{1}{F(-a^2, -ab, -b^2)} \cos(ax+by) \text{ or } \sin(ax+by)$$

$$f(-a^2, -ab, -b^2) \neq 0$$

$g(x) = \cos ax$ or $\sin ax.$

$P.I = \frac{1}{F(D)} \cos ax$ or $\sin ax$

$D^2 = -a^2$

$P.I = \frac{1}{F(-a^2)} \cos ax$ or $\sin ax$

$\overline{F(-a^2)} \neq 0$

$= 0$ case of failure

If $F(-a^2) = 0$,

$P.I = x \cdot \frac{1}{F'(-a^2)}$

if $F'(-a^2) \neq$

If $F(-a^2, -ab, -b^2) = 0$

$$P.I = \frac{1}{(bD - aD')^r} \cos(ax+by) \text{ or } \sin(ax+by).$$

$$= \frac{x^r}{b^r r!} \cancel{e^{ax+by}} \cos(ax+by) \text{ or } \sin(ax+by).$$

Note: If $f(D^2, DD', D'^2)$ cannot be transforme to the form $(bD - aD')^r$, then we can proceed as the above case I.

① Solve the following D.E ~~D^3~~

$$\left(D^3 - 6D^2D' + 11DD'^2 - 6D'^3\right) z = e^{5x+6y}$$

ans: The problem is already given in the S.F. $D^3 = D'^3$

A.E is, ~~found~~ by putting

$D = m$ & $D' = 1$.

ie, The A.E,

for solving m $m^3 - 6m^2 + 11m - 6 = 0$.

$F(m) = m^3 - 6m^2 + 11m - 6$ $\quad 1 - 6 + 11 - 6$

$f(1) = 1 - 6 + 11 - 6$

$= \underline{\underline{0}}$.

$m = 1$ is a root.

$(m-1)$ is a factor.

Then by synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$m^2 - 5m + 6 = 0$

$m = \dfrac{5 \mp \sqrt{25 - 24}}{2} = \dfrac{5+1}{2}$ or $\dfrac{5-1}{2}$

$= \dfrac{6}{2}$ or $\dfrac{4}{2}$

$\underline{\underline{3}}$ $\quad \underline{\underline{2}}$

$\underline{\underline{m = 2, 3}}$.

$\therefore$ T.F $= \phi_1(y + m_1x) + \phi_2(y + m_2x) + \phi_3(y + m_3x)$

$CF = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$

$$P.I = \frac{1}{F(D, D')} e^{ax+by}.$$

$$= \frac{1}{D^3 - 6D^2 + 11DD'^2 - 6D'^3} e^{5x+6y}.$$

$$[D = a = 5, \ D' = b = 6.]$$

$$= \frac{1}{5^3 - 6\times5^2\times6 + (11\times5\times6^2) - 6\times6^3}$$

$$= \frac{1}{\cancel{125 - 150 + 1980 - 1296 -}}$$

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$$= \frac{1}{-91} e^{5x+6y}$$

② Solve $(D - 2D')(D - D')^2 z = e^{x+y}.$

ans. $D^3 - 2D^2D' + DD'^2 - 2D^2D' + 4DD'^2 - 2D'^3$

$\overset{5}{8}F \Rightarrow (D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{x+y}$

$AE \Rightarrow D^3 = D'^3$ (power)

$\therefore D = m, \ D' = 1.$

$m^3 - 4m^2 + 5m - 2 = 0 \Rightarrow f(m) = m^3 - 4m^2 + 5m - 25$

$f(1) = 1 - 4 + 5 - 2 = 0$

$\therefore m = 1$ is a root &

$(m-1)$ is a factor

By synthetic division,

$$\begin{array}{c|cccc} 1 & 1 & -4 & 5 & -2 \\ & 0 & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

$m^2 - 3m + 2 = 0.$

$$m = \frac{3 \pm \sqrt{(-3)^2 - 4\times1\times2}}{2\times1}$$

$$= \frac{3 \pm \sqrt{1}}{2}$$

$$= \frac{3+1}{2} \ \& \ \frac{3-1}{2} = \frac{4}{2} \ \& \ \frac{2}{2}$$

$$= 2 \ \& \ 1$$

Roots are, $m = 2, 1, 1$

$$C.F = \phi_1(y+2x) + \phi_2(y+x) + x\,\phi_3(y+x).$$

$$P.I = \frac{1}{F(D,D')} e^{ax+by}.$$

$$= \frac{1}{(D-2D')(D-D')^2} e^{x+y}$$

$$[D = a = 1,\ D = b = 1]$$

$$= \frac{1}{(1-2)(1-1)^2} e^{x+y}$$

$$= \frac{1}{0} e^{x+y},$$ the case of failure.

Here converting $F(D,D')$ into the form $(bD-aD')^r$ is not simple. so we are going to the method of differentiation.

~~$P.I = x \cdot \frac{1}{(D-2D') \times 2(D-D'}$~~

$$\frac{\partial(uv)}{\partial u} = u \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial u}{\partial x}$$

$$u = D-2D'$$
$$V = (D-D')^2$$

$$P.I = x \cdot \frac{1}{\underbrace{(D-2D')}_{u} \cdot \underbrace{2(D-D')}_{v} \times 1 + \underbrace{(D-D')^2 \times 1}} e^{x+y}$$

$$[D = a = 1,\ D' = b = 1]$$

$$= x \cdot \frac{1}{(1-2)\,2(1-1)1 + (1-1)^2}$$

$$= x \cdot \frac{1}{0} e^{x+y}$$ the case of failure

$$P.I = x^2 \cdot \frac{1}{2[(D-2D')\times 1+(D-D')\times 1] + 2(D-D')\times 1} e^{x+y}$$

$$[D = a = 1, \quad D' = b = 1]$$

$$= x^2 \frac{1}{2[-1+0+0]}$$

$$= \underline{\underline{\frac{-x^2}{2} e^{x+y}}}$$

∴ The G.S,

$$z = C.F + P.I$$

$$z = \phi_1(y+2x) + \underline{\underline{\phi_2(y+x) + x\phi_3(y+x) - \frac{x^2}{2} e^{x+y}}}$$

Solve.

③ $$\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$$

Ans: The problem is already not ~~the~~ ScEq. in the S.F,

∴ The S.F is, $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$

$$D^2 = D'^2$$

A.E is, $D = m, \ D' = 1$

$$m^2 - 4m + 4 = 0$$

$$m = \frac{4 \pm \sqrt{(-4)^2 - 4\times 1\times 4}}{2\times 1}$$

$$= \frac{4\pm}{2}$$

$$= 2 \ \& \ 2$$

Case 3

Shorter Method for finding P.I when $g(x,y)$ is of the form $\phi(ax+by)$ i.e

$$g(x,y) = \phi(ax+by)$$

When $g(x,y)$ is of the form $\phi(ax+by)$

i.e, $P.I = \frac{1}{F(D,D')} \phi(ax+by).$

$= \int \cdots \iint \frac{1}{F(a,b)} \phi(v) \, dv^n.$ $\boxed{v = ax+by}$

$D = a, D' = b.$

$= \frac{1}{F(a,b)} \int \cdots \iint \phi(v) \, dv^n$

where 'n' is the degree of the homogeneous function $f(D, D')$ & after performing the integration, i.e $\phi(v)$ n times v is replaced by $ax+by$, i.e $v = ax+by$

9/4/16 **Note.**

If $F(a,b)=0$, then the RHS becomes infinite & the above formula fails. In such cases $F(D,D')$ must be factorised into factors of the for $(bD-aD')^n$. i.e our P.I becomes

$$P.I = \frac{1}{F(D,D')} \phi(ax+by)$$

$$= \frac{1}{(bD-aD')^n} \phi(ax+by)$$

$$= \frac{x^n}{n!\,b^n} \phi(ax+by)$$

1) Solve, $(D^2 + 3DD' + 2D'^2)z = 2x+3y$.

ans The problem is already in S.F.

A.E, $D^2 = D'^2$

$D = m, \quad D' = 1.$

$\therefore m^2 + 3m + 2 = 0$

$$\therefore m = \frac{-3 \pm \sqrt{9-4\times1\times2}}{2\times1}$$

$$= \frac{-3\pm1}{2} = \frac{-3+1}{2} \;\&\; \frac{-3-1}{2}$$

$$= \cancel{2} \;\&\; \cancel{1} \quad -1 \;\&\; -2$$

C.F ie

$$z = \phi_1(y-x) + \phi_2(y-2x)$$

Here, $P.I = \dfrac{1}{F(D,D')} \phi(ax+by)$

(power=2) $\dfrac{1}{F(D,D')}\phi(ax+by)$

$= \dfrac{1}{F(a,b)} \iint v\,dv\,dv$

$v = 2x+3y.$

$$= \frac{1}{D^2+3DD'+2D'^2}\phi(2x+3y)$$

~~D=a=~~

$$\left[D = a = 2,\ D' = b = 3\right]$$

$$= \frac{1}{2^2+3\times2\times3+2\times3^2}\iint v\,dv\,dx.$$

$v = 2x+3y.$

$$= \frac{1}{40}\int\frac{v^2}{2}\,dy.$$

$$= \frac{1}{80}\int v^2 dv$$

$$= \frac{1}{80}\times\frac{v^3}{3} = \frac{1}{240}v^3 = \frac{1}{240}\times(2x+3y)^3.$$

∴ the complete solution

$$z = \phi_1(y-x)+\phi_2(y-2x)+\frac{1}{240}(2x+3y)^3$$

Solve $r+s-2t = \sqrt{2x+y}$

ie case-3 $\boxed{\sqrt{\ } = \phi(ax+by)}$

Sf⇒

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-2\frac{\partial^2 z}{\partial y^2} = (2x+y)^{1/2}$$

$m = 1, -2$

AE $D^2+DD'-2D'^2 = (2x+y)^{1/2}$

$D = m,\ D' = 1$

$$m^2+m-2 = 0$$

$\sqrt{9}$

$$m = \frac{-1\pm\sqrt{1-4\times1\times-2}}{2\times1} = \frac{-1\pm3}{2}$$

$$m = \frac{-1+3}{} \ \&\ \frac{-1-3}{} = 2\ \&\ \frac{-4}{} = 1\ \&\ -2$$

ie CF ie, $z = \phi_1(y+x) + \phi_2(y-2x)$

$$P.I = \frac{1}{F(D,D')} \phi(ax+by).$$

$$= \frac{1}{D^2+DD'-2D'^2} (2x+y)^{1/2}$$

$[D = a = 2, \ D' = b = 1]$

degree = 2

$$\frac{1}{f(D,D')} \phi(ax+by)$$

$$= \frac{1}{f(a,b)} \iint \phi(v) dv dv$$

$\phi(v) = (2x+y)^{1/2}$

$\Rightarrow$ ~~v = 2x+y~~

$v = 2x+y$

$$= \frac{1}{2^2+2\times1-2\times1^2}$$ ~~$(2x+y)^{1/2}$~~ $\int\int (v)^{1/2} dv \, dv$

$\frac{1}{2}+1$

$\frac{1+2}{2}$

$$= \frac{1}{4} \int \frac{v^{1/2+1}}{1/2+1} dv$$

$3/2+1 = 5/2$

$$= \frac{1}{4} \int \frac{v^{3/2}}{3/2} dv$$

$$= \frac{2}{12} \int v^{3/2} dv = \frac{1}{6} \frac{v^{5/2}}{5/2}.$$

$$= \frac{2}{30} v^{5/2}$$

$$= \frac{2}{30} (2x+y)^{5/2}$$

$$= \frac{1}{15} (2x+y)^{5/2}$$

$\therefore$ the C.S, $z = C.F + PI$

$$z = \phi_1(y+x) + \phi_2(y-2x) + \frac{1}{15}(2x+5)^{5/2}$$

3) Solve $(D^2 - 2DD' + D'^2)z = \tan(y+x)$

AE, $D = m$, $D' = 1$

$m = 1, 1$

$m^2 - 2m + 1 = 0$

$$m = \frac{2 \pm \sqrt{4-4}}{2} = 1, \ 1$$

$$C.F = \phi_1(y+x) + x\,\phi_2(y+x)$$

$$P.I = \frac{1}{f(D,D')}\,\phi(ax+by)$$

$$= \frac{1}{D^2 - 2DD' + D'^2}\,\tan(y+x)$$

$$\frac{1}{F(D,D')}\,\phi(ax+by) = \frac{1}{f(a,b)}\iint \phi(v)\,dv\,dv$$

$$\phi(v) = \tan(y+x)$$

$$[D = a = 1,\ D' = b = 1]$$

$$= \frac{1}{0}\,\tan(y+x).$$ the case of failure

$$P.I = \frac{1}{F(D,D'}\,\tan(y+x)$$ $(D - D'^2$

$$= \frac{1}{(D-D')^2}\,\tan(y+x)$$ $\frac{x^n}{b^n \times n!}$

$(bD - aD')^n$

This $(D-D')^2 = (bD - aD')^n$ $\therefore n = 2$, $b = 1$

$$= \frac{x^2}{2! \times 1^2}\,\tan(y+x)$$ $\boxed{\frac{x^n}{n!\,b^n}\,\phi(ax+by)}$

$$= \frac{x^2}{2}\,\tan(y+x)$$

$\therefore$ C.S, i.e $z = CF + P.I.$

$$z = \phi_1(y+x) + x\,\phi_2(y+x) + \frac{x^2}{2}\tan(y+x)$$

6/4/- Module 4 Partial Differential Equation (continuing)

Method of finding P.I when $g(x,y) = x^m y^n$

As discussed in the ordinary D.E, take out the lowest degree term from $f(D,D')$ so as to reduce it in the form $[1 \pm F(D,D')]^n$

After that, take it to the numerator & get $[1 \pm F(D,D')]^{-n}$ which is then expanded with the help of binomial theorem.

If $g(x,y) = x^m y^n$, then $[1 \pm F(D,D')]^{-n}$ is expanded in powers of $\frac{D'}{D}$ if $m>n$ & expanded in powers of $\frac{D}{D'}$ if $m<n$.

1) Solve $(D^2 - 2DD' - 15D'^2)z = 12xy$

ans: A.E, $D = m$, $D' = 1$ $(m = -3, 5)$

$$m^2 - 2m - 15 = 0.$$

$$m = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times -15}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{4+60}}{2}$$

$$= \frac{2 \pm 8}{2}$$

$$= \frac{2+8}{2} \;\&\; \frac{2-8}{2}$$

$$\therefore m = \underline{\underline{5}} \;\&\; \underline{\underline{-3}}$$

$$C.F = \phi_1(y \cancel{+} -3x) + \phi_2(y+5x)$$

$$P \cdot I = \frac{1}{F(D, D')} \; x^m \; y^n \qquad \left[1 \pm F(D, D')\right]^n$$

$$= \frac{1}{D^2 - 2DD' - 15D'^2} \times 12xy \qquad x^1 y^1 = x^m y^n$$

$m = 1, \; n = 1$

$$= 12 \; \frac{1}{D^2\left[1 - \frac{2D'}{D} - 15\frac{D'^2}{D^2}\right]} \; xy$$

$$= \frac{12}{D^2} \; \frac{1}{\left[1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right]^1} \; xy \qquad \boxed{(1-x)^{-1} = 1 + x + x^2 + x^3}$$

$$= \frac{12}{D^2} \left[1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right]^{-1} xy$$

$$= \frac{12}{D^2} \left[1 + \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right] xy$$

only power, m = 1, n = 1 not 2

$$= \frac{12}{D^2} \left[xy + \frac{2D'}{D} xy + \cancel{\frac{15D'^2}{D^2} xy}\right]$$

$\underline{\underline{0}}$

D' = w.r.to y $= \underline{\underline{x}}$

$$= \frac{12}{D^2} \left[xy + \frac{2}{D} x + \frac{15}{D^2} \times 0\right]$$

$$\frac{15D'^2}{D^2} xy = D'^2(xy) = D'(x) = \underline{\underline{0}}$$

$$= 12 \left[\frac{1}{D^2}(xy) + \frac{2}{D^3}(x)\right]$$

$$\frac{1}{D} xy = \frac{x^2}{2} y$$

$$\frac{1}{D}\left(\frac{x^2}{2}\right)$$

$$= 12 \left[\frac{1}{D} \frac{x^2}{2} y + \frac{2}{D^2}\left(\frac{x^2}{2}\right)\right]$$

$$= 12 \left[\frac{x^3}{6} y + \frac{2}{D}\left(\frac{x^3}{6}\right)\right] = 12$$

$$= 12\left[\frac{x^3}{6}y + 2\frac{x^4}{\underset{12}{24}}\right]$$

$$= \frac{12}{6}x^3y + \frac{x^4}{12} \times 12$$

$$= \underline{\underline{2x^3y + x^4}}$$

Friday
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2) Find the reab function v of x & y, reducing to zero when $y = 0$. and satisfying.

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = -4\pi\,(x^2+y^2).$$

Ans: Here the real function V satis fying the given equation can be obtained by finding the P. I of the given eq. equation.

The symbolic of the given equation is

$$(D^2 + D'^2)\,v = -4\pi\,(x^2+y^2)$$

$$P.I = \frac{1}{F(D, D')}\, g(x, y) = x^m y^n$$

$m < n$

$\frac{D}{D'}$

$x^2y^0 + y^2 \cdot x^0$

$$= \frac{1}{D^2+D'^2}\; -4\pi\,(x^2+y^2)$$

$\left[1 \pm F(D, D')\right]^{1}$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$

$n = 2\ (x^2, D^2, D'^2)$

$$= -4\pi \frac{1}{D^2\left[1+\frac{D'^2}{D^2}\right]}\,(x^2+y^2)$$

$$= \frac{-4\pi}{D^2} \times \left[1+\frac{D'^2}{D^2}\right]^{-1} \times (x^2+y^2)$$

$$= \frac{-4\pi}{D^2}\left[1 - \frac{D'^2}{D^2}\right]\quad (x^2+y^2)$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2) - \frac{D'^2}{D^2}(x^2+y^2)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2) - \frac{1}{D^2}*(2)\right]$$

$$= \frac{-4\pi}{D^2}\left[(x^2+y^2) - \frac{1}{D}\cdot\frac{1}{D}(2)\right]$$

$$= -\frac{4\pi}{D^2}\left[(x^2+y^2) - 2\frac{1}{D}x\right]$$

$$= -\frac{4\pi}{D^2}\left[(x^2+y^2) - 2\frac{x^2}{2}\right]$$

$$= -\frac{4\pi}{D^2}\left[(\cancel{x^2}+y^2 - \cancel{x^2})\right]$$

$$= \left[-\frac{4\pi}{D^2}\, y^2\right]$$

$$= \left(-4\pi\,\frac{1}{D}\left(\frac{1}{D}y^2\right)\right)$$

$$= -4\pi\,\frac{1}{D}y^2 x.$$

$$= -4\pi\; y^2\frac{x^2}{2}$$

$$= -2\pi\,\cancel{y}\,x^2y^2$$

$$\therefore V = -2\pi x^2y^2$$

with reduces ~~to~~ zero
when $y=0$,

$$V = -2\pi \times x^2 \times 0 = 0$$

$D'($

$D' =$

$D'^2 =$

$D'^2(x$

$\frac{1}{D}$

$\frac{1}{D} =$

finding The method of Finding P.I
when $g(x,y)$ is of the form $e^{ax+by}\cdot V(x,y)$

$$g(x,y) = e^{ax+by}\cdot V(x,y).$$

$$P.I = \frac{1}{F(D,D')} e^{ax+by}\cdot V(x,y).$$

ordinary D.E
$g(x) = e^{ax}\cdot V$
$P.I = \frac{1}{F(D)}\left(e^{ax}\cdot V\right)$
$PI = e^{ax}\cdot \frac{1}{F(D+a)} V.$

$$P.I = e^{ax+by}\cdot \frac{1}{F(D+a, D'+b)} V(x,y)$$

Solve $(D^2 + DD' - 6D'^2)z = y \sin x.$

$SF \Rightarrow D^2 + DD' - 6D'^2.$

$D = m, \quad D' = 1$ $\qquad m = 2, -3$

$\therefore A.E \Rightarrow m^2 + m - 6 = 0$

$$m = \frac{-1 \pm \sqrt{1^2 - 4\times1\times-6}}{2\times1}$$

$$= \frac{-1 \pm \sqrt{\cancel{24}\,1+24}}{2} = \frac{-1+5}{2} \ \& \ \frac{-1-5}{2}$$

$$= \frac{4}{2} \ \& \ -\frac{6}{2} = 2 \ \& \ -3$$

C.F, $z = \phi_1(y+2x) + \phi_2(y-3x)$

$$P.I = \frac{1}{F(D,D')} g(x,y).$$

$$P \cdot I = \frac{1}{F(D,D')} g(x,y)$$

$$= I \cdot P . \text{of } e^{ix} xy$$

$$\sin x \cdot x y$$

$e^{ix} = \cos x + i \sin x$

$\sin x$ = Imaginary part of $e^{i x}$

i.e,

$\sin x = I \cdot P \cdot \text{of } e^{ix}$

$$P \cdot I = \frac{1}{D^2 + DD' - 6D'^2} \; I \cdot P \text{ of } e^{ix} xy$$

Note:

$e^{ix} = \cos x + i \sin x$

$\cos x$ = Real Part of e^{ix}

i.e $\cos x = R \cdot P \text{ of } e^{ix}$

$$= I \text{ P of } e^{ix+0y} \frac{1}{(D+i)^2 + (D+i)(D'+0) - 6(D'+0)^2} xy$$

(ix)

D changed to D + i

D' ,, to D' + 0 (0y)

$$= I\text{P of } e^{ix+0y} \frac{1}{D^2 + 2Di + i^2 + (DD' + 0 + D'i + 0) - 6D'^2} xy$$

$$= I\text{P of } e^{ix} \frac{1}{D^2 + 2iD - 1 + DD' + iD' - 6D'^2} xy$$

$$= I \cdot P \text{ of } e^{ix} \frac{1}{(D^2 + DD' - 6D'^2) + (2iD + iD') - 1} xy$$

No need to

* $y = y^1 x^0$

$m = 0, n = 1$

$m < n$

$\hookrightarrow \frac{D}{D'}$

const

$$= I \cdot P \text{ of } e^{ix} \frac{1}{D^2 + DD' - 6D'^2 + i(2D + D') - 1}$$

Multiplying by using [illegible] (minus)

$$(1 \pm F(D,D'))^{n}$$

$$= -I \cdot P \text{ of } e^{ix} \frac{1}{1 - [i(2D + D') + D^2 + DD' - 6D'^2]}$$

$$= -I \cdot P \text{ of } e^{ix} \left[1 - (i(2D + D') + D^2 + DD' - 6D'^2)\right]^{-1}$$

$= -\text{I.P of } e^{ix}\left[1+i(2D+D')+D^2+DD'-6D'^2\right]y$ $\boxed{(1-x)^{-1} = 1+x+x^2.}$

$= -\text{I.P of } e^{ix}\left[y+i(2D(y)+D'(y))\right]$

$= -\text{IP of } e^{ix}\left[y+i(1)\right]$ $\boxed{e^{ix} = \cos x + i\sin x}$

$= -\text{I.P of } \left[(\cos x + i\sin x)\ (y+i)\right].$

$= -\text{I.P of } \left[y\cos x + i\cos x + i\sin x \times y + i^2\sin x\right]$

$\text{of } \left[y\cos x + i\cos x + y\sin x - \sin x\right]$

$= -(y\sin x + \cos x)$

Solve $\left(D^2 + DD' - 2D'^2\right)z = (y-1)e^{x}.$

A.E, $D = m,\ D' = 1.$

$m^2 + m - 2 = 0$

$m = \dfrac{+1 \pm \sqrt{1^2 - 4\times 1\times -2}}{2\times 1}$ $\quad m = -1, 2.$

$= \dfrac{+1 \pm \sqrt{1+8}}{2}$

$= \dfrac{+1+3}{2}$ & $\dfrac{+1-3}{2} = \dfrac{4}{2}$ & $\dfrac{-2}{2}$ $= 2 \,\&\, -1$

C.F ie $z = \phi_1(y-x) + \phi_2(y+2x).$

$\text{P.I} = \dfrac{1}{F(D,D')} e^{ax+by} \cdot V(x,y).$

$= \dfrac{1}{D^2 + DD' - 2D'^2}(y-1)e^{x}.$ $\quad D \to D+1$, $D' \to D'+0$

$= e^{1x+0y} \dfrac{1}{(D+1)^2 + (D+1)(D'+0) - 2(D'+0)^2}(y-1).$

$$= e^{x} \frac{1}{D^2+2D+1^2 \pm DD' \pm 0 \pm D'+0-2D'^2}(y-1)$$

$$= e^{x} \cdot \frac{1}{D^2+1+2D-DD'-D'-2D'^2}(y-1)$$

$$= e^{x} \cdot \frac{1}{\left[1+(2D-D'-DD'+D^2-2D'^2)\right]^{1}}(y-1)$$

$$= e^{x} \left[1 + (2D-D'-DD'+D^2-2D'^2)\right]^{-1}(y-1)$$

$$= e^{x} \left[1 - \underset{\to 0}{\cancel{2D}} + D' + \cancel{DD'} + \cancel{D^2} + \cancel{2D'^2}\right](y-1)$$

$D'(y-1) = 1 \quad (0)1 = 0$

$\boxed{(1+x)^{-1} = 1-x+x^2-x^3+\ldots}$

$(y-1)$

$\boxed{\text{Power of } y = 1}$

$$= e^{x} (1 + D')(y-1).$$

D' w.r.to y

$\to D'(y-1) = 1$

$$= e^{x}\left[(y-1) + \underset{1}{D'(y-1)}\right]$$

$$= e^{x}(y-1+1)$$

$$= e^{x}(y)$$

$$= ye^{x}$$

c.s i.e

$$Z = \phi(y-x) + \phi_2(y+2x) + ye^{x}.$$

✓ (3) Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 3e^{2x} - 2x^2y$

Ans

- $P.I = \frac{1}{F(D,D')} f(x,y) = \frac{1}{D^3-2D^2D'}\left(\underset{(PI_1)}{3e^{2x}} - \underset{(PI_2)}{2x^2y}\right)$
- $PI_1 = \frac{1}{D^3-2D^2-D'} 3e^{2x+0y}$, • $PI_2 = \frac{1}{D^3-2D^2D'}(2x^2y)$ $m=3, n=1$ $m>n \Rightarrow \frac{D'}{D}$

$\boxed{\text{As done in Tutorial of Module}}$

④ Solve $(D^2 D' - 2DD'^2 + D'^3) z = \frac{1}{x^3}$

$D' = m, \quad D = 1.$

A E ⇒

$$m^3 - 2m^2 + m = 0 \qquad m = 0, 1, 1$$

$$m(m^2 - 2m + 1) = 0$$

$$m = 0, \qquad m^2 - 2m + 1 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 4 \times 1 \times 1}}{2 \times 1}$$

$$m = \frac{2}{2} \;\&\; \frac{2}{2} \Rightarrow \underline{\underline{m = 1, 1}}.$$

C.F is $z = \phi_1(y) + \phi_2(y+x) + x\phi_3(y+x)$.

$$P.I = \frac{1}{F(D, D')} g(x, y).$$

$$= \frac{1}{D^2 D' - 2DD'^2 + D'^3} \left(\frac{1}{x^3}\right)$$

$$= \frac{1}{\underline{D'}(D^2 - 2DD' + D'^2)} \left(\frac{1}{x^3}\right) \qquad \frac{1}{D'}\left(\frac{1}{x^3}\right) = \underline{\underline{\frac{1}{x^3} \times y}}$$

$$= \frac{1}{D^2 - 2DD' + D'^2} \left(\frac{y}{x^3}\right).$$ [illegible]

$$= \frac{1}{(D - D')^2} \left(\frac{y}{x^3}\right)$$

$$\frac{1}{D - mD'} g(x, y) = \int g(x, c - mx)$$

Remaining c

$= 1 \quad 1 \times (y)$

$$= \frac{1}{(D-D')} \int \left(\frac{c-x}{x^3} \right) dx.$$

$$\boxed{\begin{matrix} D = m \\ y = c - x \end{matrix}}$$

5) Sol

$$= \frac{1}{(D-D')} \int \left(\frac{c}{x^3} - \frac{1}{x^2} \right) dx.$$

$$* \int c \frac{1}{x^3} dx$$

$$c \int x^{-3} dx = \frac{x^{-2}}{-2}$$

$$= \frac{-1}{2x^2}$$

$$= \frac{1}{(D-D')} \quad c \cdot \log x \left(\frac{-c}{2x^2} - -\frac{1}{x} \right)$$

$$* \int \frac{1}{x^2} dx =$$

$$\int x^{-2} dx = \frac{x^{-1}}{-1}$$

$$\frac{-2+1}{-2+1} \frac{-1}{1} = -\frac{1}{x}$$

$$= \frac{1}{D-D'} \left[-\frac{c}{2x^2} + \frac{1}{x} \right]$$

$$c = y + mx$$

$$\Rightarrow c = y + x$$

$$= \frac{1}{D-D1} \left[\frac{-(y+x)}{2x^2} + \frac{1}{x} \right]$$

(D - mD') case

$$\text{y change to} \Rightarrow \boxed{\begin{matrix} m = 1 \\ y = c' - x \end{matrix}}$$

$$= \int \frac{-(c' - x + x)}{2x^2} + \frac{1}{x} dx.$$

$$-c \int \frac{1}{2x^2} dx = -\frac{c}{2} \int x^{-2}$$

$$= \frac{-c}{2} \times \frac{x^{-1}}{-1}$$

$$= \frac{c}{2} \times \frac{1}{x}$$

$$x^{-1+1}$$

$$= \int \frac{-c'}{2x^2} + \frac{1}{x} dx$$

$$= \frac{c'}{2x} + \log x$$

Remann

$$c = y + mx.$$

$$c' = y + x.$$

$$= \frac{y+x}{2x} + \log x = \frac{y}{2x} + \frac{1}{2} + \log x$$

∴ The C.S is $z = \phi_1(y) + \phi_2(y+x) + \phi_3(y+x) + \frac{y}{2x} + \frac{1}{2} + \log x.$

(5) Solve $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (2x^2 + xy - y^2)\sin xy - \cos xy$

ans) A.F ⇒ $(D^2 - DD' - 2D'^2)z = 0$

A.E ⇒ $m^2 - m - 2 = 0$ $(m = -1, 2)$

$$m = \frac{1 \pm \sqrt{1 - 4 \times 1 \times -2}}{2 \times 1}$$

$$= \frac{1 \pm 3}{2}$$

$$= \frac{1+3}{2} \ \& \ \frac{1-3}{2}$$

$$= \underline{\underline{2}} \ \& \ \underline{\underline{-1}}$$

c.f ie $z = \phi_1(y - \underline{\underline{x}}) + \phi_2(y + 2x)$.

$$P.I = \frac{1}{F(D, D')} g(x,y).$$

$$= \frac{1}{D^2 - DD' - 2D'^2} (2x^2 + xy - y^2) \sin xy - \cos xy$$

general case

$\frac{1}{D - mD'} g(x,y) = \int g(x, c - mx)$

$c \to y + mx$

$$= \frac{1}{(D - 2D')(D + D')} [(2x - y)(x + y)] \sin xy - \cos xy$$

$D - mD'$ $m = -1$

$c = y + mx$

$c = y - x \Rightarrow y = \underline{\underline{c + x}}$

$D - mD'$

$D + D' \Rightarrow \underline{\underline{m = -1}}$

$y = c - mx$

$y = \underline{c + x}$

$$= \frac{1}{D - 2D'} \int [2x - (c + x)(x + c + x) \sin x(c + x) - \cos x(c + x)]$$

$$= \frac{1}{D - 2D'} \int [(x - c)(2x + c). \underbrace{\sin(cx + x^2)} - \cos(cx + x^2)]$$

$$= \frac{1}{D-2D'} \int \left[(x-c) \left(-\frac{d}{dx} \cos(cx+x^2) \right) \right] dx -$$

$\int uv\,dv$ $\quad \int \cos(cx+x^2)\,dx.$

$$= \frac{1}{D-2D'} \left\{ (x-c)\left(+\cos(cx+x^2)\right) \right\} +$$

$\boxed{\int u\,dv = uv_1 - u'v_2 + u''}$

$$\boxed{\int \cos(cx+x^2)\,dx} - \cancel{\int \cos(cx+x^2)\,dx.}$$

$$= \frac{1}{D-2D'} (c-x) \cos(cx+x^2)$$

$c = y+mx$
$c = y+x$
$y = c-x$
$(m=-1)$

$$= \frac{1}{D-2D'} (y-x-x) \quad \cos\left((y-x)x + x^2\right)$$

$$= \frac{1}{D-2D'} (y-2x) \quad \cos(xy - \cancel{x^2} + \cancel{x^2})$$

$y = y+mx.$
$y = y+2x$
$y = c$

$y = c - mx$

$$= \frac{1}{D-2D'} (y-2x) \quad \cos xy$$

$D - m D'$

$y = c - 2x$

$$= \int (c'-2x-2x) \cos x\cancel{y}(c'-2x)\,dx.$$

$$= \int \underbrace{(c'-4x)\cos(c'x - 2x^2)}_{\text{derivative of sin}}\,dx.$$

$$= \int \frac{d}{dx} \sin(c'x - 2x^2)\,dx.$$

$\boxed{c' = y+2x}$

$$= \sin(c'x - 2x^2)$$

$$= \sin((y+2x)x - 2x^2) =$$

Module - 5

One Dimension ition.

A Partial Differential Equation is an equation that contains one or more partial derivative of an unknown function that depends on atleast two variable. P.D.E(s) are very important in Fluid dynamics, elasticity, electromagnetic theory & Quantum mechanics. In this chapter we realize an important P.D.E arising in physics Wave eqn of the vibrating elastic string.

In this chapter we are discussing about the method for obtaining solutions to P.D.E(s) subjects to the given conditions.

The method of Separation of Variables.

Here we will discuss the most commonly used method ie, the method of separation of variables for obtaining the solution to the P.D.E(s) subjects to the given condition.

Suppose that given PDE, contains 'n' independent variable $x_1, x_2, \cdots x_n$ and o dependent variable 'u'. In the method of separation of variables, we assume that 'u' can be expressed as

$u(x_1, x_2 \ldots\ldots x_n) = X_1(x_1) \times X_2(x_2) \times \cdots\cdots X_n(x_n)$ -

where 'X_i' is a function of 'x_i' only. ie, $i = 1, 2, \ldots\ldots n$.. Substituting the ① in th given P.D.E and if we get 'n' ordinary D.E one in each of the unknown functio X_i which are then solved for X_i. then the PDE can be solved using the method o separation of variables.

(1) Solve $x \cdot \frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0$, by using the method of separation of variables.

Ans:

$$x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0 \quad \text{———} ①$$

Let $u(x,y) = X_{(x)}\, Y_{(y)}$ —— ②

$$\frac{\partial u}{\partial x} = X'_{(x)}\, Y_{(y)} = X'Y$$

ie, $\frac{\partial u}{\partial y} = X_{(x)}\, Y'_{(y)} = XY'$

Substituting this $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in ①

$$x X'Y - 2yXY' = 0$$

$\therefore \frac{xX'}{X} = \frac{2yY'}{Y}$ — ④ LHS is a function of x only

Here L.H.S is a function of x only and R.H.S is a function of y only. Since x & y are independent variable. The above eqn ④ is true if each side is equal to same constant (which is called the separation constant).

ie, $\frac{xX'}{X} = \cancel{y}k$ & $\frac{2yY'}{Y} = k.$

$$\frac{xX'}{X} = k. \qquad X' = \frac{dX}{dx}.$$

$$x\,\frac{dX}{dx.\,X} = k.$$

$$\cancel{x}\,\frac{dX}{X} = k\,\frac{dx}{x}.$$

$$\to \log X = k \log x + \log c_1$$

$$X = c_1 x^{k}.$$

$$\therefore \frac{2yY'}{Y} = k \qquad Y' = \frac{dY}{dy}$$

$$\Rightarrow \frac{2y\,dY}{Y\cancel{y}\,dy} = k.$$

$$\to \frac{dY}{Y} = \frac{k}{2}\cdot\frac{dy}{y}.$$

$$\log Y = \frac{k}{2}\log y + \log c_2$$

$$Y = c_2\, y^{k/2}$$

Hence the general solution is $U = XY$

$$\Rightarrow U = c_1 x^{k} \cdot c_2 y^{k/2} = Cx^{k}y^{k/2} \quad \boxed{c_1 \times c_2 = C}$$

11|1L16

(2) Solve $\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$ 5

By using the method of separation of variables

ans:- The given P.D.E contain x & y. as independent variables & 'u' as a dependent variable.

∴ The given problem has a solution of the form $X(x), \& Y(x)$ where X is a function of x only & Y is a function of y only. $u(x,y) = X(x)\, Y(y)$ ——①

Then differenting the ① partially w.r. to x, we get $\frac{\partial^2 u}{\partial x^2} = X''Y$

$$\frac{\partial u}{\partial x} = X'Y$$

$$\frac{\partial u}{\partial y} = XY'$$

where '$'$'(s) denotes the derivative w.r. to the relevant variable. Substituting these derivatives in the given eqn. we get

$$X''Y - 2X'Y + XY' = 0$$

on separating variables [After dividing the above eqn throughout by XY]

$$\frac{X'' - 2X'}{X} + \frac{Y'}{Y} = 0$$

$$\frac{X'' - 2X'}{X} = -\frac{Y'}{Y} \quad \text{——②}$$

Since x, y are independent variables.

$\therefore$ the above eqn is true, if and only if each side is equal to the same constant say 'k'

k = constant.

$$\frac{X'' - 2X'}{X} = \frac{-Y'}{Y} = k \quad \text{——} ③$$

which gives,

$$\frac{X'' - 2X'}{X} = k \quad \Rightarrow \quad X'' - 2X' = kX$$

$$X'' - 2X' - kX = 0 \quad \text{——} ④$$

This is an O.D.E

SF is, $(D^2 - 2D - k)X = 0$

$$\cancel{\lambda} = \frac{2 \pm \sqrt{4 - 4 \times 1 \times -k}}{2 \times 1} = \cancel{\frac{2 \pm 2\sqrt{2}}{2}} \; \frac{2 \pm \sqrt{4 + 4k}}{2}$$

$$= \frac{2 \pm \sqrt{4(1+k)}}{2} = \frac{2 \pm \sqrt{4}\sqrt{1+k}}{2}$$

$$= 1 \pm \sqrt{1+k}$$

$$\cancel{= 1 \pm \sqrt{2} = 2.414 \;\&\; -0.414}$$

$$\begin{array}{r|l} 2 & 8 \\ 2 & 4 \\ \hline & 2 \end{array}$$

A.E is, $\lambda^2 - 2\lambda - k = 0$.

$$\lambda = 1 \pm \sqrt{1+k}$$

$$\lambda = 1 + \sqrt{1+k}, \; 1 - \sqrt{1+k}$$

$$\text{C.F} = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

Thus the solution ④ is

ie $$X = c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x} \quad \text{——} ⑤$$

Again from ③ $\Rightarrow$

$$-\frac{Y'}{Y} = k, \quad \cancel{ie}$$

$$-Y' = kY \Rightarrow \cancel{-\frac{dY}{dy} = kY}$$

$$\frac{-dY}{dy \times Y} = k$$

$$\Rightarrow \frac{dY}{Y} = -k\,dy \Rightarrow \log Y = -ky + \cancel{a} \log c_3$$

$$Y = c_3 e^{-ky} \quad \text{——} ⑥$$

∴ the solution of Equation ①
ie, the general solution, ie

$$u(x,y) = \left(c_1 e^{(1+\sqrt{1+k})x} + c_2 e^{(1-\sqrt{1+k})x}\right) c_3 e^{-ky}$$

$$u(x,y) = \left[A e^{(1+\sqrt{1+k})x} + B e^{(1-\sqrt{1+k})x}\right] e^{-ky}$$

where $c_1 c_3 = A$, $c_2 c_3 = B$.

A, & B are the arbitary constants.

(3) Use the method of separation of variables
a.

$$\frac{c^2 \partial^2 u}{\partial x^3} = \frac{\partial u}{\partial t}$$ which tends to zero as $t \to \infty$ $\forall x$. $u \to 0$

The eqn is called the 1st dimension heat-eqn

ans: The given PDE is $\frac{c^2 \partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$ ——①

Using the method of separation of variable the given eqn has the solution of the form

$$u(x,t) = X(x)\,T(t) = XT \quad \text{——②}$$

Indepe variable = x,t
dependent = u

By diff. the ② w.r.to the independent variable x & t we get $\frac{\partial u}{\partial t} = XT'$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

where the '(s) represents derivative

Substituting the above values in ① we get

$$c^2 X'' T = X T' \quad \text{———} ③$$

Now arguing exactly as in the previous problem we have.

Dividing $c^2 X'' T = XT' = k.$

~~Then~~ Now we are separating the variable after dividing XT.

$$\frac{c^2 X''}{X} = \frac{T'}{T} = k. \quad \text{———} ④$$

The above eqn ④ gives

$$\frac{c^2 X''}{X} = k.$$

$$c^2 X'' = X k$$

$$c^2 X'' - kX = 0$$

$$X'' - \frac{k}{c^2} X = 0 \quad \text{———} ⑤$$

$$\frac{T'}{T} = k.$$

$$T' = kT$$

$$T' - kT = 0 \quad \text{———} ⑥$$

Now for, obtaining the solution satisfying the given boundary condition the constant 'k' may have the following 3 possibility

(a) If $k = 0$ in case I

$$X'' = 0$$

$$\text{Si} = D^2 X = 0$$

$$\text{AE} \rightarrow \lambda^2 = 0 \qquad \text{(i.e.)}\ \lambda = 0, 0$$

$$\text{c.s,}\ X = (c_1 + c_2 x) e^{0x} = \underline{\underline{c_1 + c_2 x}} \text{ or } \underline{\underline{c_1 x + c_2}}$$

Case I If $T' = 0$

$$T = c_3 e^{0x} \Rightarrow \underline{\underline{T = c_3}}$$

(b) If k is positive.

ie, $k = +p^2$

$$X'' - \frac{p^2}{c^2} X = 0$$

SF $\left(D^2 - \frac{p^2}{c^2}\right)x = 0$

AE = $\lambda^2 - \frac{p^2}{c^2} = 0$

$$\lambda^2 = P^2/c^2 \Rightarrow \boxed{\lambda = \pm \frac{P}{c}}$$

$$X = c_4 e^{(P/c)x} + c_5 e^{(-P/c)x}.$$

$$T' - \frac{p^2}{\cancel{c^2}} T = 0$$

$$T' - p^2 T = 0$$

$$(D - p^2)T = 0$$

$$\lambda - p^2 = 0$$

$$\underline{\underline{\lambda = p^2}}$$

ie $\underline{\underline{T = c_6 e^{p^2 t}}}$

c) If 'k' is negative

~~$k = +p^2$~~

$$\boxed{k = -p^2}$$

$$X'' + \frac{p^2}{c^2} X = 0$$

SF⇒

AE⇒ $\lambda^2 + p^2/c^2 = 0$

$$\lambda^2 = -\frac{p^2}{c^2}$$

$$\lambda = \pm\sqrt{P^2/c^2}$$

$$\lambda = \pm i\frac{P}{c}$$

CF $\alpha = 0$,

$$X = C_7 \cos\left(\frac{P}{c}\right)x + C_8 \sin\left(\frac{P}{c}\right)x$$

$c_7\ c_8\ c_9$.

$$T' + p^2 T = 0$$

$$\lambda + p^2 = 0$$

$$\lambda = -p^2$$

$$\underline{\underline{T = c_9 e^{-p^2 t}}}$$

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ie In this case our solution

$$u(x,t) = X(x)\, T(t) = XT.$$

$$u(x,t) = \left(C_7 \cos\left(\frac{p}{c}\right)x + C_8 \sin\frac{p}{c}x\right] C_9 e^{-p^2t.} \quad ⓒ$$

$$u(x,t) = (c_1 x + c_2)c_3 \quad —— ⑦$$

$$u(x,t) = \left[c_4 e^{(p/c)x} + c_5 e^{(-p/c)x}\right] c_6 e^{p^2t} \quad ⑧$$

$$u(x,t) = \left[c_7 \cos\left(\frac{p}{c}\right)x + c_8 \sin\left(\frac{p}{c}\right)x\right] c_9 e^{-p^2t} \quad ⑨$$

Hence from the eqn ⓐ the solution corresponding to the above 3 possibilities are given by eqn ⑦, ⑧ & ⑨

In these three solutions we have to consider that solution which satisfy the boundary condition $u \to 0$ when $t \to \infty$ for all x. Clearly the solution given by eqn ⑨ corresponding to $\underline{k = -p^2}$ is the required solution.

(4) Solve the boundary value problem

$$\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0$$ satisfying the condition

$z(x,0) = 0$, $z(x,\pi) = 0$, z

$z(0,y) = 4 \sin 3y$

ans: where x & y → independ variables

z → dependent variables.

$$\underline{z(x,y) = X(x)\,Y(y) = XY} \quad — ②$$

Give, $\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2} = 0$ ——①

using the method of separation of variable the general solution of eqn ① has the form is given by ②

Here X is a function of 'x' only & Y is a function of 'y' only, substituting the partial derivatives w.r.to x & y of ② in ① we get

$$X'Y + XY'' = 0$$

Then by dividing the above eqn throughout by XY, & separating variables we get

$$\frac{X'}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X'}{X} = \frac{-Y''}{Y} = k \quad ③$$

The given form of boundary conditions show that Y(y) must be of trignometric form. To satisfy this requirement the separation constant $\underline{k}$ in ③ is taken as $\underline{p^2}$ (given condi y = +ve then the eqn (9) give;

$$\frac{X'}{X} = k$$

$$X' = KX$$

$$\therefore X' - KX = 0 \text{ —— } (4)$$

$$X' - p^2 X = 0$$

$$AE \Rightarrow \lambda - p^2 = 0.$$

$$\therefore \underline{\underline{\lambda = p^2}}$$

C.F ⇒

$$\underline{\underline{X = c_1 e^{p^2 x}}} \text{ —— } ⑥$$

$$\frac{-Y''}{Y} = K$$

$$-Y'' = KY$$

$$Y'' + KY = 0 \text{ —— } ⑤$$

$$Y'' + p^2 Y = 0.$$

$$AE \Rightarrow \lambda^2 + p^2 = 0.$$

$$\lambda^2 = -p^2$$

$$\lambda = \pm\sqrt{-p^2}.$$

$$(\alpha = 0) \quad \underline{\underline{\lambda = \pm ip}}.$$

$$\underline{\underline{Y = c_2 \cos py + c_3 \sin p}} \text{ —— } ⑦$$

∴ From ② the solution of the given P.D.E is given by

$$\underline{\underline{z(x, y) = c_1 e^{p^2 x_1}\left[c_2 \cos py + c_3 \sin py\right]}} \text{ —— } ⑧$$

Using the first boundary condition
conditions

* $z(x, 0) = 0$, we get.

$$z(x, 0) = c_1 e^{p^2 x}\left[c_2 \times 1 + 0\right]$$

$$\Rightarrow c_1 e^{p^2 x} \times c_2 = 0.$$

i.e, $\underline{C_2 = 0}$ & $c_1 e^{p^2 x} \neq 0$

Condition 2

- $z(x, \pi) = 0$, we get

$z(x,\pi) = c_1 e^{p^2x} [c_2 \cos p\pi + c_3 \sin p\pi]$

From condition 2, $\underline{C_2 = 0.}$

ie, $z(x,\pi) = c_1 e^{p^2x} [0 \times \cos p\pi + C_3 \sin p\pi]$

$z(x,\pi) = c_1 e^{p^2x} \times C_3 \sin p\pi$

$\Rightarrow c_1 e^{p^2x} \times C_3 \sin p\pi = 0$

ie $\therefore \sin p\pi = 0$, & $C_3 \neq 0.$

$\sin p\pi = \sin n\pi$

$p\pi = n\pi$

$\therefore \underline{p = n}$

~~From~~ In eqn ⑧ substitute, $\underline{C_2 = 0 \text{ \& } p = n}$

$z(x,y) = c_1 e^{n^2x} [0 \times \cos ny + c_3 \sin ny]$

$z(x,y) = c_1 e^{n^2x} \times c_3 \sin ny$

$z(x,y) = c_1 c_3 e^{n^2x} \sin ny$ ——— ⑨

Using the principle of superposition the eqn ⑨ can be taken as

$z(x,y) = \sum_{n=1}^{\infty} A_n e^{n^2x} \sin ny$ ——— ⑩

where ($\underline{\underline{A_n = c_1 c_3}}$)

onditions

$z(0, y) = 4 \sin 3y.$ we get

$(x=0, y=y)$

ie, $4 \sin 8y = \sum_{n=1}^{\infty} A_n e^{n^2 x 0} \sin ny.$

$$4 \sin 3y = \sum_{n=1}^{\infty} A_n \sin ny.$$

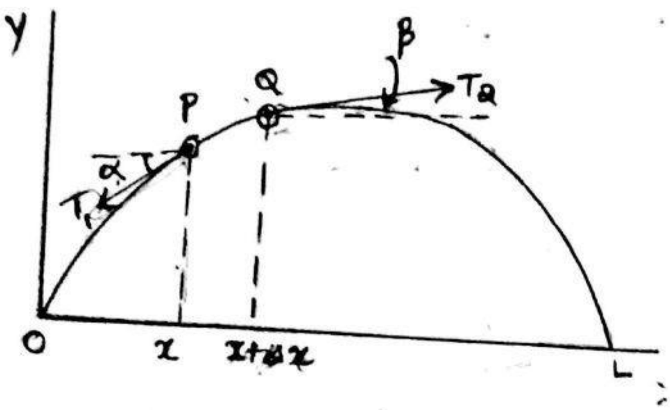
where $A_n = \begin{cases} 4, & n = 3 \\ 0, & n \neq 3 \end{cases}$

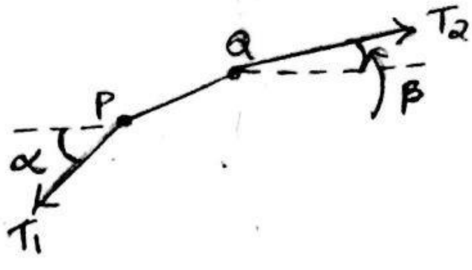
Wave Equation.

The wave equation is $\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \cdot \frac{\partial^2 u}{\partial x^2}}.$

ie $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}}$

where $\boxed{c = \sqrt{\frac{T}{M}}}$





Consider a tightly stretched elastic string of length l with its end points fixed. Let the string be released from rest and allowed to vibrate. The problem is to find the deflection $y(x,t)$ at any point x and at any time $t>0$. For deriving the partial differential equation we make the following assumptions.

1. The mass of the string per unit length is constant.
2. The string is perfectly elastic and does not offer any resistant to bending.
3. The tension caused by stretching the string before fixing it at the end points is so large compared to the weight of the string and hence the action of the gravitational force can be neglected.
4. The string performs a small transverse vibration in a vertical plane. In other words, every particle of the string moves strictly vertically so that the deflection and the slope at any every point of the string remains small in absolute value.

imp

Take one end of the string as origin O and the x-axis along the string. We assume that th motion takes place entirely in the xy plane. Consi the forces acting on a small portion PQ of the string where $P(x,y)$ and $Q(x+\Delta x, y+\Delta y)$.

Since the string does not offer resistence to bending the tension at each point of the string is tangential to the curve of the string. Let T_1 and T_2 be the tensions at the points P and Q. Let α and β be the angles made by tangents at P and Q respectively with the X-axis. Since the points of the string move vertically, there is no motion in the horizontal direction. Hence the horizontal components of the tension must be constant.

$$\therefore T_1 \cos\alpha = T_2 \cos\beta = T \text{ constant}.$$

Let m be the mass per unit length of the string. The vertical component of the force acting on the elemant PQ is

$$T_2 \sin\beta - T_1 \sin\alpha.$$

Accelaration of the elemant in the y-direction is

$$\frac{\partial^2 y}{\partial t^2}.$$

Then by Newtons second law of motion.

$$T_2 \sin\beta - T_1 \sin\alpha = m\,\Delta x \,\frac{\partial^2 y}{\partial t^2}. \quad \text{———} (1)$$

Dividing (1) by $T_1\cos\alpha = T_2\cos\beta = T$, we get

$$\frac{T_2\sin\beta}{T_2\cos\beta} - \frac{T_1\sin\alpha}{T_1\cos\alpha} = \tan\beta - \tan\alpha = \frac{m\Delta x}{T}\frac{\partial^2 y}{\partial t^2} \quad -(2)$$

Let $\tan\alpha$ and $\tan\beta$ are the slopes of the strings at P and Q.

$$\therefore \tan\alpha = \left(\frac{\partial y}{\partial x}\right)_P \text{ and } \tan\beta = \left(\frac{\partial y}{\partial x}\right)_Q.$$

$$\therefore (1) \Rightarrow \left(\frac{\partial y}{\partial x}\right)_Q - \left(\frac{\partial y}{\partial x}\right)_P = \frac{m\Delta x}{T}\frac{\partial^2 y}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\left[\left(\frac{\partial y}{\partial x}\right)_Q - \left(\frac{\partial y}{\partial x}\right)_P\right]}{\Delta x}$$

Taking limits as $\Delta x \to 0$ we get

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

$$\therefore \frac{\partial^2 y}{\partial t^2} = a^2\frac{\partial^2 y}{\partial x^2}, \text{ where } a^2 = \frac{T}{m} \quad -(3)$$

This partial differential equation is known as the one dimensional wave equation.

In this equation, we have

$$c = \sqrt{\frac{T}{m}}.$$

Here, c has the dimensions of length/time and hence the same dimensions as a velocity. In order to determine the subsequent behaviour of the string, the displacement $u(x,t)$ must satisfy certain boundary conditions.

[illegible] to fixed supports thus giving the two boundary conditions (BC):

$$BC : \begin{cases} u(0,t) = 0 \\ u(\underset{l}{a},t) = 0 \end{cases} \quad (t \geq 0) \text{ --- } (4)$$

In addition, since the partial derivative u_u that appears in the wave equation is of the second order, we must specify two initial conditions (IC) which are the initial position (or initial displacement) and the initial velocity of the string. These conditions are written as

$$IC : \begin{cases} u(x,0) = f(x) \\ \frac{\partial u(x,0)}{\partial t} = u_t(x,0) = g(x) \end{cases} \quad (0 \leq x \leq a) \text{ --- } (5)$$

where $f(x)$ and $g(x)$ represant the given initial position and velocity respectively. The problem is to find the solution of the partial differential Equations that also satisfies the conditions ④ & (5). Such a problem is called an initial-boundary value problem

of the one dimensional wave
on using the method of separation
able

---

ume that the solution of wave eqn

$$\frac{{}^{2}u}{\partial x^{2}} = c^{2}\frac{\partial^{2}u}{\partial t^{2}} \text{ ——— ① } \quad \text{is}$$

in the separated form as

$$) = X(x)\cdot T(t) \text{ ——— ②}$$

bstituting the derivatives of eqn ② and separating variables by the
the separation constant 'k'

$$''T = c^{2} X T''$$

g the above eqn throughout by XT.
separating the variables

$$= c^{2}\frac{T''}{T} = k \text{ ——— ③}$$

here we get two ordinary eqn

$$'' = k \qquad\qquad \frac{c^{2}T''}{T} = K$$

$$'' = kX \qquad\qquad c^{2}T'' = KT$$

$$-KX = 0 \text{ ——— ④} \qquad\qquad c^{2}T'' - KT = 0.$$

$$T'' - \frac{K}{c^{2}}T = 0. \text{ ——— ⑤}$$

If we can solve these two O.D.E, we can multiply their solutions together Obtaining the general solution.

ie, $U_{(x,t)} = X_{(x)} T_{(t)}$ of wave equation, the form of the solution changes for $K<0$, $K=0$ & $K>0$

Now we can solve the eqn ④ & ⑤ by using the above three possibilities.

* When $K=0$

$X'' = 0$

AE ⇒ $\lambda^2 = 0$

$\lambda = 0,0$

$X = c_1 + c_2 x$ $\boxed{\text{or}}$

$X = c_1 x + c_2$.

$T'' = 0$

$\lambda^2 = 0$

$\lambda = 0,0$

$T = c_3 + c_4 t$ (OR)

$c_3 t + c_4$

* When K is negative $(K<0)$

ie $K = -p^2$

$X'' + p^2 X = 0$

$\lambda^2 + p^2 = 0$

$\lambda^2 = -p^2$

$\lambda = \pm\sqrt{-p^2}$

$T'' + \frac{p^2}{c^2} T = 0$

$\lambda^2 + \frac{p^2}{c^2} = 0$

$\lambda^2 = -p^2/c^2$

$\lambda = \pm\sqrt{-p^2/c^2} = \pm i\, p/c$

c.f. $X = \underline{C_5 \cos px + C_6 \sin px}$

$T = \underline{\underline{C_7 \cos\left(\frac{p}{c}\right)t + C_8 \sin\left(\frac{p}{c}\right) t}}$.

* when $\underline{k \text{ is positive}}$ $(k>0)$.

$\underline{k = +p^2}$

$X'' - p^2 X = 0$

$\lambda^2 - p^2 = 0$

$\lambda^2 = p^2$

$\lambda = \pm p$.

c.f. $\underline{\underline{X = C_9 e^{px} + C_{10} e^{-px}}}$

$T'' - \frac{p^2}{c^2} T = 0$

$\lambda^2 - \frac{p^2}{c^2} = 0$

$\lambda^2 = \frac{p^2}{c^2}$

$\lambda = \pm \frac{p}{c}$.

$\underline{\underline{T = C_{11} e^{(p/c)t} + C_{12} e^{-(p/c)t}}}$

The solutions of the given eqn.

$u(x,t) = (C_1 x + C_2)(C_3 t + C_4)$ ——— ⑥

$u(x,t) = (C_5 \cos px + C_6 \sin px)\left(C_7 \cos\left(\frac{p}{c}\right)t + C_8 \sin\left(\frac{p}{c}\right)t\right)$ ⑦

$u(x,t) = \left(C_9 e^{px} + C_{10} e^{-px}\right)\left(C_{11} e^{(p/c)t} + C_{12} e^{-(p/c)t}\right)$ ⑧

Here the solutions are not satisfying the condition when $\underline{k = -p^2}$.

imp

* So we are taking wave eqn $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$

Then by the method of separation of variable we get

$u(x,t) = X(x)\,T(t)$

The above eqn. $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$ ——(i)

becomes

$$X'' T = \frac{1}{c^2} X T''$$

Then by the method of separation of variable & by dividing each side by XT and arrange we get

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = K \text{ ——(ii)}$$

from (ii)

$\frac{X''}{X} = k$

$X'' = kX$

$X'' - kX = 0$ ——(iii)

$\frac{1}{c^2} \frac{T''}{T} = k$

$\frac{T''}{T} = kc^2$

$T'' - kc^2 T = 0$

——(iv)

then we can solve the eqns (iv) & (iii) by using 3 possibilities $K<0$, $K=0$, $K>0$.

case 1

* $\underline{K<0}$ (k is negative)

$k = -p^2$

$X'' + p^2 X = 0$

$\lambda^2 + p^2 = 0$

$\lambda^2 = -p^2$

$\lambda^2 = \sqrt{-p^2}$

$\lambda = \pm ip$

$T'' + c^2 p^2 T = 0$

$\lambda^2 + c^2 p^2 = 0$

$\lambda^2 = -c^2 p^2$

$\lambda = \sqrt{-c^2 p^2}$

$\lambda = \pm icp$

$\therefore X = C_1 \cos px + C_2 \sin px$ | $T = C_3 \cos cpt + C_4 \sin cpt$

case 2

* when $k = 0$

$$X'' = 0$$

$$\lambda^2 = 0$$

$$\lambda = 0, 0$$

$$X = C_5 x + C_6$$

$$T'' = 0$$

$$T^2 = 0$$

$$T = 0, 0$$

$$T = C_7 t + C_8$$

* case 3

when $k > 0$ (k is +ve

$$k = p^2$$

$$X'' - p^2 X = 0$$

$$\cancel{X'' = p}$$

$$\lambda^2 - p^2 = 0$$

$$\lambda^2 = p^2$$

$$\lambda = \pm p$$

$$X = C_9 e^{px} + C_{10} e^{-px}$$

$$T'' - p^2 c^2 T = 0$$

$$\lambda^2 - c^2 p^2 = 0$$

$$\lambda^2 = c^2 p^2$$

$$\lambda = \pm cp$$

$$T = C_{11} e^{cpt} + C_{12} e^{-cpt}$$

$\therefore$ The solution of the wave eqn

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$ become

$k = -p^2$ is (V)

$$u(x,t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad \text{(V)}$$

$$u(x,t) = (C_5 x + C_6)(C_7 t + C_8) \quad \text{(VI)}$$

$$u(x,t) = (C_9 e^{px} + C_{10} e^{-px})(C_{11} e^{cpt} + C_{12} e^{-cpt}) \quad \text{(VII)}$$

Now out of these 3 solution we have to choose that solution which is consistant with the physical nature of the problem.

As we are discussing the problem of vibration the displacement $u(x,t)$ must contain the period functions of 't'.

Hence, the $u(x,t)$ must contain the trignometric term. Thus the solution of the eqn is the first solution. ie.

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

The solution of the wave eqn.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \times \frac{\partial^2 u}{\partial t^2} \quad \text{is}$$

$$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)}$$

with an intial ~~d~~ displacement, $u(x,0) = F(x)$.

solution:

The displacement $u(x,t)$ of the string is described by the one dimensional wave equation ; * $\frac{\partial^2 u}{\partial x^2} = \frac{1}{C^2} \cdot \frac{\partial^2 u}{\partial t^2}$ —— ①

The boundary condition of fixed end may be written as

* $u(0,t) = 0$ —— ②

* $u(L,t) = 0$ —— ③

~~and~~ And since the string is released from the rest the initial velocity is zero.

and hence * $\frac{\partial}{\partial t} u(x,0) = 0$ —— ④.

The initial displacement is F(x) then we have * $u(x,0) = F(x)$ —— ⑤.

The general solution of the equation ① by the method of separation of variables is

* $u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$ —— ⑥

Applying the first ~~the~~ boundary condition $u(0,t)=0$

i.e equation ② in eqn ⑥, we get.

$0 = c_1 [c_3 \cos cpt + c_4 \sin cpt]$

$$\Rightarrow u(x,t) = (c_2 \sin px)\left[c_3 \cos cpt + c_4 \sin cpt\right] \longrightarrow ⑦$$

Now we are applying the second boundary condition i.e $u(L,t) = 0$) i.e the eqn ③ substitute in eqn ⑦

$$0 = c_2 \sin pL\left[\underbrace{c_3 \cos cpt + c_4 \sin cpt}_{\Rightarrow \neq 0}\right]$$

$c_2 \neq 0$ (because c_1 & c_2 are arbitrary constan $c_2 \neq 0$]

$\Rightarrow^{*}$ $\sin pL = 0.$ & $c_2 \neq 0$

$\Rightarrow \sin pL = \sin n\pi$

i.e, $pL = n\pi$

* i.e $p = \dfrac{n\pi}{L}$ (eigen value) ⟶ ⑧

Here 'p' is called an eigen value when 'p' is substituted in 'sin px' then the function '$\sin \frac{n\pi x}{L}$' is called an eigen function. Then

Each eigen value corresponding to a particular value of 'n' is a unique eigen function. It should be noted that the eigen value for 'n' = 0 is excluded here as a possible solution, Since, this corresponds to $p = 0$

Now we are substituting the value $P = \frac{n\pi}{L}$ in eqn

$$u(x,t) = \left(C_2 \sin \frac{n\pi x}{L}\right)\left[C_3 \cos \frac{cn\pi t}{L} + C_4 \sin \frac{cn\pi t}{L}\right]$$

so, by the method of superposition principle we get ~~u(x,t)=~~

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos \frac{cn\pi}{L}t + B_n \sin\left(\frac{n\pi}{L}\right)\right] \times \left(\sin \frac{n\pi}{L} x\right)$$

when $A_n = C_2 C_3$ & $B_n = C_2 C_4$.

e,

$$(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos \frac{n\pi c}{L}t + B_n \sin \frac{n\pi c}{L}t\right] \times \sin \frac{n\pi x}{L} \quad ⑨$$

Inorder to apply the condition ~~for~~ ④, we differentiate eqn ⑨ with respect to 't'

e,

$$\frac{\partial}{\partial t}(u(x,t)) = \sum_{n=1}^{\infty}\left[A_n \times -\sin \frac{n\pi c}{L}t \times \frac{n\pi c}{L} + B_n \cos \frac{n\pi c t}{L} \times \frac{n\pi c}{L}\right] \times \left(\sin \frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[A_n \times -\sin \frac{n\pi c}{L}t \times \cancel{\frac{n\pi}{L}} + B_n \cos \frac{n\pi c}{L}t \times \cancel{\frac{n\pi}{L}}\right] \times \sin\left(\frac{n\pi x}{L}\right)$$

sing the condition for in the above eqn

$$\frac{\partial}{\partial t}u(x,0) = 0$$

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[0 + B_n\right] \times \sin\left(\frac{n\pi x}{L}\right)$$

$$0 = \sum_{n=1}^{\infty} \underbrace{\frac{n\pi c}{L}}_{\neq 0} B_n \underbrace{\sin \frac{n\pi x}{L}}_{\neq 0}$$

$* \therefore B_n = 0$, $\frac{n\pi c}{L} \neq 0$, $\sin \frac{n\pi x}{L} \neq 0$

Hence from ⑨ [applying $B_n=0$]

ie

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\frac{n\pi ct}{L} \cdot \sin\frac{n\pi x}{L} \longrightarrow ⑩$$

Finally the constant A_n is determined by the use of the 2nd boundary condition ie is the equation ⑤ which gives

$$u(x,0) = F(x) \quad —⑤$$

Apply ~~t=0~~

$$F(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

which represents the half range Fourier sine series for $F(x)$ where the coefficient ~~A_n~~ =

$$A_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx.$$

18/4/16
Monday

(2)Q Find the displacement of $u(x,t)$, if the string is given an initial displacement $f(x)$ and an initial velocity $g(x)$?

Ans: The displacement $u(x,t)$ the string is given by the 1 Dimensional eqn

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad —①$$

Since the end point of the string are fixed, we have the boundar condition as

$u(0,t) = 0$ —②

$u(L,t) = 0$ —③

The initial displacement is $f(x)$

$u(x,0) = f(x)$ —④

and the initial velocity is $g(x)$ ie,

$\frac{\partial}{\partial t} u(x;0) = g(x)$ —⑤

using the method of separation of variable the solution of eqn ⑤ is

$u(x,t) = (c_1 \cos px + c_2 \sin px)\ (c_3 \cos cpt + c_4 \sin cpt)$ —⑥

By applying the boundary condition 2 in eqn 6, we get $c_1 = 0$

By applying the boundary condition 3 in eqn 6, we get $p = \frac{n\pi}{L}$, (where $n = 1,2,3..$)

Substituting these 2 values in Eqn⑥ and using the principle of superposition we get a general solution as

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos \frac{n\pi c}{L} t + B_n \sin \frac{n\pi c}{L} t\right] \times \sin \frac{n\pi x}{L}$$ —⑦

Now using the initial condition ~~5~~ 4 in eqn ⑦

$$F(x) = \sum_{n=1}^{\infty} \left[A_n \times 1 + B_n \times 0\right] \sin \frac{n\pi x}{L}.$$

$$F(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$$

clearly this is a Halfrange Fourier sine series. where A_n, is

$$A_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx. \qquad ⑧$$

For finding B_n, now we are going ~~Now we as~~ to apply the initial conditions 5, In eqn ⑦

$$\frac{\partial}{\partial t} \overset{\text{condition 5}}{u(x,0)} = g(x) \qquad ⑤$$

For that purpose first we are differentiate the eqn ⑦ partially w. ~~r~~ to 't'.

$$\frac{\partial(u(x,t))}{\partial t} = \sum_{n=1}^{\infty} \frac{n\pi c}{L} \left(A_n\left(-\sin \frac{n\pi c}{L} t\right) + B_n \cos \frac{n\pi c}{L} t\right)\right)$$

$$* \sin\left(\frac{n\pi x}{L}\right)$$

Now we are applying the eqn ⑤ in the above eqn.

$t=0$

$$g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L}\left[A_n \times 0 + B_n \times 1\right] \sin\frac{n\pi x}{L}$$

$$g(x) = \sum_{n=1}^{\infty} B_n \times \frac{n\pi c}{L} \times \sin\frac{n\pi x}{L}.$$

clearly this is a Half range fourier sine series. where B_n is,

$$B_n \cdot \frac{n\pi c}{K} = \frac{2}{L}\int_0^L g(x) \times \sin\frac{n\pi x}{L}\, dx. \quad \cancel{⑨}$$

$$\therefore B_n = \frac{2}{n\pi c}\int_0^L g(x) \times \sin\left(\frac{n\pi x}{L}\right) dx \quad ⑨$$

Hence the required solution is obtained as eqn⑦ where A_n & B_n are given by eqn ⑧ & eqn ⑨.

③ A tightly stretched Homogeneous string of length 'L', with its ends fixed at $x=0$, & $x=L$. Execute transverse vibration. Motion starts with zero initial velocity by displacing the string into the form $F(x) = K(x^2 - x^3)$ Find the deflection $u(x,t)$ at in it time 't'.

ans: The deflection $u(x,t)$ of the vibrating string is given by the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad \text{——} ①$$

Since, the string is stretched b/w the fixed point $x = 0$ & $x = L$

The boundary conditions are

$$u(0,t) = 0 \quad \text{——} ②$$

$$u(L,t) = 0 \quad \text{——} ③$$

The motion starts with a zero initial velocity. So, the initial condition become

$$\frac{\partial}{\partial t} u(x,0) = 0 \quad \text{——} ④$$

The displacement $F(x)$ is given by

$F(x) = k(x^2 - x^3)$, so that

$$u(x,0) = k(x^2 - x^3) \quad \text{——} ⑤$$

By the method of separation of variable the general solution of eqn ① is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\,(c_3 \cos cpt + c_4 \sin cpt) \qquad \text{⑥}$$

Now we are applying the boundary conditions 2 & 3 in eqn ⑥:

we get $c_1 = 0$ & $p = \frac{n\pi}{L}$ $(n = 1, 2, 3 \ldots)$

Now we are applying the above two conditions in eqn ⑥ and applying the principle of Super position. we get, the general solution of eqn ① as.

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi ct}{L} + B_n \sin \frac{n\pi ct}{L}\right] \times \sin \frac{n\pi x}{L} \qquad \text{⑦}$$

Refer Qus1)

For applying the eqn ④ in eqn ⑦ First we are partially differentiating the eqn ⑦ with respect to 't' and applying the initial condition 4 we get,

$$\underline{B_n = 0},$$

Hence. eqn 7 becomes:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi c}{L} t \times \sin \frac{n\pi x}{L} \qquad \text{⑧}$$

Now we are applying the eqn ⑤ in eqn ⑧. we get

$$t=0$$

$$F(x) = k(x^2 - x^3) = \sum_{n=1}^{\infty} A_n \times 1 \times \sin\frac{n\pi x}{L}.$$

i.e,

$$F(x) = k(x^2 - x^3) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L}$$

clearly this is a Halfrange ~~Fourier~~ Fourier sine series where A_n is

$$A_n = \frac{2}{L}\int_0^L f(x)\,\sin\left(\frac{n\pi x}{L}\right)dx.$$

$$A_n = \frac{2}{L}\int_0^L k(x^2-x^3) \times \sin\frac{n\pi x}{L}\,dx.$$

$$A_n = \frac{2k}{L}\int_0^L (\underset{u}{x^2 - x^3})\ \underset{v\,dx}{\sin\left(\frac{n\pi x}{L}\right)}\,dx$$

Furda, $uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$

$$= \frac{2k}{L}\left[\left((x^2-x^3) \times \frac{-\cos\frac{n\pi x}{L}}{n\pi/L}\right)_0^L - \left((2x-3x^2) \times \frac{-\sin\frac{n\pi x}{L}}{(n\pi/L)^2}\right)_0^L + \left((2-6x) \times \frac{\cos\frac{n\pi x}{L}}{(n\pi/L)^3}\right)_0^L - \left(-6 \times \frac{\sin\frac{n\pi x}{L}}{(n\pi/L)^4}\right)_0^L\right]$$

$$= \frac{2k}{L}\left[-\left((x^2-x^3)\frac{\cos\frac{n\pi x}{L}}{n\pi/L}\right)_0^L + \left((2x-3x^2)\frac{\sin\frac{n\pi x}{L}}{(n\pi/L)^2}\right)_0^L + \left((2-6x)\frac{\cos\frac{n\pi x}{L}}{(n\pi/L)^3}\right)_0^L - \left[-6 \times \frac{\sin}{}\right.\right.$$

$$= \frac{2k}{L}\left[\left(-(L^2-L^3)\frac{\cos n\pi}{n\pi/L}\right)+\left((2L-3L^2)\frac{\sin n\pi}{(n\pi/L)^2}\right)\right.$$

$$+\left(2-6L\frac{\cos n\pi}{(n\pi/L)^3}-2\times\frac{1}{(n\pi/L)^3}\right)$$

$$\left.+\left(6\frac{\sin n\pi}{(n\pi/L)^4}\right)\right.$$

$$= \frac{2k}{L}\left[(L^2-L^3)\left(\frac{-L}{n\pi}\cos n\pi\right)+0+(2-6L)\frac{L^3}{n^3\pi^3}\cos n\pi - \frac{2L^3}{n^3\pi^3}+0\right]$$

$$= \frac{2k}{L}\left[\cancel{L^2-L^3}\ \frac{-1\times L}{n\pi}(L^2-L^3)\cos n\pi+\frac{L^3}{n^3\pi^3}\left((2-6L)\cos n\pi-2\right)\right]$$

$$A_n = 2k\left[\frac{(-1)^{n+1}}{n\pi}(L^2-L^3)+\frac{La}{n^3\pi^3}\left[(2-6\cancel{L})((-1)^n-2)\right]\right] \quad \text{⑨}$$

Here the require solution is given by ⑥ where A_n is given by eqn ⑨

④ As tightly stretched violin of length 'a' and fixed at both end. is plucked at its midpoint & assume initially the shape of a triangle of Height 'h' as shown in the figure. Find the displacement $u(x,t)$ at any distance 'x' and any time 't' after the string is released from rest.

Ans

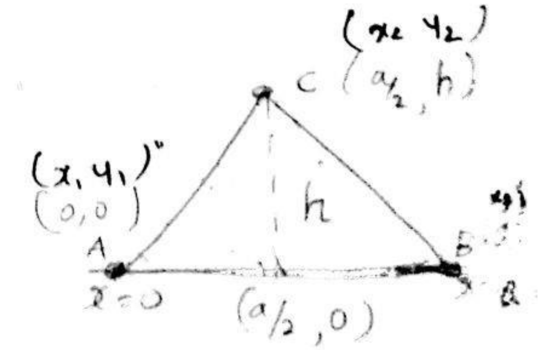
The displacement $u(x,t)$ of the string is given by the 1 dimensional wave eqn $$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \quad ①$$

Since the string is tightly stretched b/w A and B with length a.
Then our boundary conditions are

$$u(0,t) = 0 \quad ②$$
$$u(a,t) = 0 \quad ③$$

since initially the string is released from rest

$$\frac{\partial}{\partial t}\left(u(x,0)\right) = 0 \quad ④$$



Equation of AC, is

$$\frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2}$$

$$\frac{x - 0}{0 - a/2} = \frac{y - 0}{0 - h}$$

$$\frac{x}{-a/2} = \frac{y}{-h} \Rightarrow \frac{2x}{a} = \frac{y}{h}$$

$$y = \frac{2xh}{a} \qquad 0 < x < a/2$$

Equation of CB,

$$\frac{x - x_2}{x_2 - x_3} = \frac{y - y_2}{y_2 - y_3}$$

$$\frac{x - a/2}{a/2 - a} = \frac{y - h}{h - 0} \Rightarrow \frac{x - a/2}{a/2 - a} = \frac{y - h}{h}$$

$$\frac{\frac{2x - a}{2}}{\frac{a - 2a}{2}} = \frac{y - h}{h}$$

$$\frac{2x - a}{a - 2a} = \frac{y - h}{h} \Rightarrow 2xh - ah = ya + 2ah - ha - 2ay$$

$$2xh - \cancel{ah} + 2ah + \cancel{ah} = -ay$$

$$\therefore y = \frac{2h}{a}(a - x), \quad a/2 < x < a. \quad \boxed{\frac{2h}{a}(a-x)}$$

$$u(x,0) = f(x) = \begin{cases} \frac{2hx}{a}, & 0 < x < a/2 \\ \frac{2h(a-x)}{a}, & a/2 < x < a \end{cases} \qquad ⑤$$

By the method of separation of variable general solution of eqn ① is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt) \qquad ⑥$$

Applying the boundary condition ⓐ and ⓑ in eqn ⑥ we get two condition

$$c_1 = 0, \; \& \; p = \frac{n\pi}{L} = \frac{n\pi}{a} \quad (n = 1, 2, 3 \cdots)$$

Now substituting the above two conditions, eqn ⑥ and applying the principle of superposition we get.

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi c}{a} t + B_n \sin \frac{n\pi c}{a} t \right] \times \sin \frac{n\pi x}{a} \quad \text{⑦}$$

Now we are going to apply eqn ④ in eqn ⑦

For that purpose first we have to take the partial derivative of eqn ⑦ w.r.to 't' applying eq ④.

$$\underline{\underline{B_n = 0}}$$

Now applying the condition $B_n = 0$ in eq ⑦ we get

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos \frac{n\pi c}{a} t \times \sin \frac{n\pi x}{a} \quad \text{⑧}$$

Now we are going to apply the initial conditions in eqn ⑧

$$u(x,0) = f(x) = \begin{cases} \frac{2hx}{a}, & 0 < x < a/2 \\ \frac{2h}{a}(a-x), & a/2 < x < a. \end{cases}$$

(t=0)

$$= \sum_{n=1}^{\infty} A_n \times 1 \times \sin\frac{n\pi x}{a}$$

So, clearly this is a function Halfrange sine series. where,

$$A_n = \frac{2}{a}\int_0^a f(x)\sin\frac{n\pi x}{a}\,dx.$$

$$A_n = \frac{2}{a}\left[\int_0^{a/2}\underset{\downarrow u}{\frac{2hx}{a}}\,\underset{\downarrow v}{\sin\frac{n\pi x}{a}}\,dx + \int_{a/2}^{a}\underset{\downarrow u}{\frac{2h}{a}(a-x)}\,\underset{\downarrow v}{\sin\frac{n\pi x}{a}}\,dx\right.$$

$$= \frac{2}{a}\left[\left(\frac{2hx}{a}\times\frac{-\cos\frac{n\pi x}{a}}{n\pi/a}\right)_0^{a/2} - \left(\frac{2h}{a}\times\frac{-\sin\frac{n\pi x}{a}}{(n\pi/a)^2}\right)_0^{a/2}\right] +$$

$$\left[\left(\frac{2h}{a}(a-x) - \frac{\cos\frac{n\pi x}{a}}{n\pi/a}\right)_{a/2}^{a} - \left(\frac{2h}{a}(\cancel{a}-1)\times-\frac{\sin\frac{n\pi x}{a}}{(n\pi/a)^2}\right)_{a/2}^{a}\right]$$

$$= \frac{2}{a}\left[-\left(\frac{2hx}{a}\times\frac{\cos\frac{n\pi x}{a}}{n\pi/a}\right)_0^{a/2} + \left(\frac{2h}{a}\times\frac{\sin\frac{n\pi x}{a}}{(n\pi/a)^2}\right)_0^{a/2}\right] +$$

$$\left[-\left(\frac{2h}{a}(a-x)\frac{\cos\frac{n\pi x}{a}}{n\pi/a}\right]_{a/2}^{a} + \left(\frac{2h}{a}(\cancel{a}-1)\,\frac{\sin\frac{n\pi x}{a}}{(n\pi/a)^2}\right]_{a/2}^{a}\right]$$

$$= \frac{4h}{a}\left[x \frac{\left(-\cos\frac{n\pi x}{a}\right)}{(n\pi/a)} + \frac{\sin\frac{n\pi x}{a}}{(n\pi/a)^2} \right]_0^{a/2} +$$

$$\left((a-x)\left(\frac{-\cos\frac{n\pi x}{a}}{(n\pi/a)}\right) + (-1)\frac{\sin\frac{n\pi x}{a}}{\left(\frac{n\pi}{a}\right)^2} \right)_{a/2}^{a}$$

$$= \frac{4h}{a^2}\left[\frac{a}{2}\left(\frac{-a}{n\pi}\cos\frac{n\pi}{2}\right) + \frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2} \right] -$$

$$\left[\frac{a}{2} \times \left(-\frac{a}{n\pi} \times \cos\frac{n\pi}{2}\right) - \frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2} \right]$$

$$= \frac{4h}{a^2}\left(2\frac{a^2}{n^2\pi^2}\sin\frac{n\pi}{2} \right)$$

$$= \frac{8h}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$\Rightarrow A_n = \begin{cases} \frac{8h}{n^2\pi^2} \sin\frac{n\pi}{2}, & n \text{ is odd} \\ 0, & n \text{ is even.} \end{cases}$$

and 'u' is the dependent variable Then by substituting above partial derivatives in eqn ① we get

$$X''T = \frac{1}{h} * XT'$$

By dividing the above eqn throughout by 'XT' and arranging

LHS as a functions of x only &

RHS as a function of 't' only

& above set is called ~~fault~~ if and only if each side is equal to the same constant 'k'

$$\frac{X''}{X} = \frac{1}{h}\frac{T'}{T} = k \text{ —— } ②$$

From here we get two O.D.E(s) such as,

$$\frac{X''}{X} = k \qquad \& \qquad \frac{1}{h}\frac{T'}{T} = k$$

$$X'' - kX = 0 \text{ —— } ③$$

$$T' - khT = 0 \text{ —— } ④$$

As we discussed the wave equation the constant 'k' may have the following three possibilities.

(a) If $k = 0$

$X'' = 0.$ $\qquad\qquad$ $T' = 0$

AE ⇒ $\lambda^2 = 0$ $\qquad\qquad$ ∴ AE, $\lambda = 0$

$\therefore \lambda = 0, 0$

C.F.,

$X = (c_1 x + c_2) e^{0x}$ $\qquad\qquad$ $\lambda = c_3 e^{0t}$

$\boxed{X = c_1 x + c_2}$ $\qquad\qquad$ $\boxed{\lambda = c_3}$

i.e. Here $u(x,t) = (c_1 x + c_2) \times c_3.$

which shows that the solution is independent of time. i.e. not possible, hence we reject this solution

(b) If K is positive

i.e. $K = p^2$ $\qquad$ & $\quad T' - p^2 hT = 0.$

$X'' - p^2 X = 0.$ $\qquad\qquad$ AE, ~~$T' = hp^2 T$~~

AE,

$\lambda^2 - p^2 = 0$ $\qquad\qquad$ $\lambda - p^2 h = 0$

$\lambda^2 = p^2$ $\qquad\qquad$ $\lambda = p^2 h.$

$\lambda = \pm p.$

$\boxed{X = c_1 e^{px} + c_2 e^{-px}}$ $\qquad\qquad$ $\boxed{T = c_3 e^{hp^2 t}}$

∴ Here $u(x,t) = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{hp^2 t}$

which shows that the solution has unbounded temperature for large t due to the exponential growth. Again this is not possible. physically

(c) If k is negative

i.e $k = -p^2$

$X'' + p^2 X = 0$

AE ⇒

$\lambda^2 + p^2 = 0$

$\lambda^2 = -p^2$

CF,

$\lambda = \pm i p.$

$\alpha = 0$

CF,

$X = (c_1 \cos px + c_2 \sin px)$

$T' + hp^2 T = 0$

~~$T' = h$~~

AE ⇒

$\lambda + hp^2 = 0$

$\lambda = -(hp^2)$

$\therefore$ CF, $T = c_3 e^{-(hp^2)t}$

$\therefore$ The General solution

$u(x,t) = (c_1 \cos px + c_2 \sin px) \cdot c_3 e^{-(hp^2)t}$

which shows that the temperature decreases as 't' increases which is correspond to the physical nature of this problem.

$\therefore$ The G.S of the one dimensional heat transfer eqn is selected as

$$\boxed{u(x,t) = (c_1 \cos px + c_2 \sin px)\ c_3 e^{-hp^2 t}}$$

* Transient state u-temp decreases, t-increases

If we compare the one dimensional Heat Transfer equation with the

Solving these problem, we find that the heat conduction equation contain only a first derivative $\frac{\partial u}{\partial t}$ with respect to the time while the wave eqn contain the second derivative $\frac{\partial^2 u}{\partial t^2}$. This means that the heat eqn require only one initial temperature condition $u(x,0)$ to determine the temperature $u(x,t)$.

Q (30) A long insulated rod with ends at zero temperature.

Let us consider a long rod of length 'L' which given initial temperature distribution along its axis & let the ends of the rod we kepted zero temperature The lateral surface of the rod is insulated which prevents the heat flow in the radial direction. and hence the temperature will depend on x-coordinate only. In this case the problems of Heat conduction is to solve the heat transfer-equation i.e,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h}\frac{\partial u}{\partial t} \quad -①$$

under the boundary condition $u(0,t) = 0$ & $u(L,t) = 0$ for every t and the initial condition is

$u(x,0) = F(x) \quad \forall x$

Note: The boundary conditions with zero end temperatures are known as Homogeneous boundary conditions

Qus:
(1) Find the temperature distribution on a rod of length 'L' whose end points are fixed at temperature zero and the initial temperature distribution is $f(x)$.

Ans: The temperature distribution $u(x,t)$ on a rod is given by the one dimensional heat eqn

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \times \frac{\partial u}{\partial t} \quad \text{——} ①$$

Here We are considering the rod of length 'L' with the end points are fixed So, our boundary conditions are

$$u(0,t) = 0 \quad \text{——} ②$$

$$u(L,t) = 0 \quad \text{——} ③$$

The initial temperature is $f(x)$. i.e the initial condition $u(x,0) = f(x)$ —— ④

using the method of separation of variable the solution of ① is taken as

$$u(x,t) = (C_1 \cos px + C_2 \sin px) \, C_3 e^{-hp^2 t} \text{ --- } ⑤$$

Now we are applying the first boundary condition · ie eqn ② in eqn ⑤.

$$0 = [C_1 \times 1 + C_2 \times 0] \, C_3 e^{-hp^2 t}.$$

$$\therefore C_1 \cdot C_3 e^{-hp^2 t} = 0$$

$$\Rightarrow \boxed{C_1 = 0} \quad \& \quad C_3 e^{-hp^2 t} \neq 0$$

Now we are applying the above obtained condition, $C_1 = 0$ in eqn ⑤

$$u(x,t) = C_2 C_3 \sin px \times e^{-hp^2 t}$$

$$u(x,t) = B_n \sin px \times e^{-hp^2 t} \text{ --- } ⑥ \qquad \boxed{C_2 C_3 = B_n}$$

Now we are applying the 2nd boundary condition · ie eqn ③ in eqn ⑥

$x = L$

$$0 = B_n \sin pL \times e^{-hp^2 t} \qquad e^{-hp^2 t} \neq 0$$

$$\sin pL = 0$$

$$\Rightarrow \sin pL = \sin n\pi$$

$$pL = n\pi \quad \Rightarrow \quad \boxed{P = \frac{n\pi}{L}} \quad n = 1, 2, 3 \ldots$$

where, L = length of the rod

$n = 1, 2, 3 \cdots$

Now Applying the above obtained condition $p = \frac{n\pi}{L}$, in = 1, 2, 3.. in eqn ⑥ and also applying the principle of superposition we get;

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \times e^{-h \frac{n^2\pi^2}{L^2} t} \longrightarrow ⑦$$

Now we are applying the initial temperature condition. i.e, the eqn ④ in the above eqn ⑦

$t = 0$,

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \times e^{-h \frac{n^2\pi^2}{L^2} \times t}$$

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \times 1$$

i.e, $f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$

which is clearly a halfrange fourier Sine Series

$$\therefore B_n = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L} \, dx \longrightarrow ⑧$$

Here the required temperature distributio is given by the eqn ⑦ and B_n is giv by the eqn. ⑧

A long insulated rod with ends at non-zero temperature.

If the end point of a rod 'R' maintain at any non-zero temperature u_0, then we reduce their end temperature to zero by defining a new variable $\boxed{u_1 = u - u_0}$, so that $\overline{u_1 = 0}$ at both ends, we then find a solution for $u_1(x,t)$ where desired temperature is given by.

$$\boxed{u(x,t) = U_0(x) + u_1(x,t)}$$

Note:

- The boundary condition with non-zero temperatures are called non-homogeneous boundary condition

① A long iron rod of ~~length~~ with insulated latteral surface has its left end maintained at a temperature ~~[illegible]~~ 0 °c and its right end at $x = 2$, maintained at 100 °c.

Determine the temperature as a func of 'x' and 't'. If the initial temperat is $u(x,0) = \begin{cases} 100x, & 0 < x < 1 \\ 100; & 1 < x < 2 \end{cases}$

Ans: Here, the initial temperature function is given.

ie, Our initial condition is

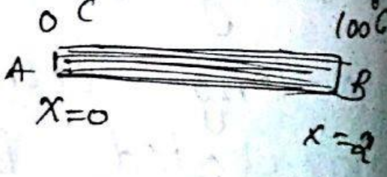
$$u(x,0) = \begin{cases} 100x, & 0 < x < 1 \\ 100, & 1 < x < 2 \end{cases} \quad \text{——} ①$$

Now, the rod of length '2' is fixed at both ends and left side is having a temperature 0°C and right side is having a temperature 100°C. So, the boundary conditions are

$$\left.\begin{array}{l} u(0,t) = 0 \quad \& \\ u(L,t) = 0 \end{array}\right\} \text{——} ②$$

$$u(2,t) = 100.$$

(Figure)



These boundary condition corresponding t the case k = 0 in the derivation of solution of one dimenssional heat equation. which gives

$u(x,t) = (C_1 x + C_2) C_3$

Now, first we are applying the given boundary condition. i.e,

$u = 0$, when $x = 0$ &

$u = 100$ when $x = 2$

[where • u = temperature, • t = time].

$0 = (C_1 \times 0 + C_2) C_3$

i.e $\boxed{C_2 C_3 = 0}$ —— Ⓐ

$100 = (C_1 \times 2 + C_2) C_3$

$\therefore 2 C_1 C_3 + C_2 C_3 = 100$ —— Ⓑ;

In eqn Ⓑ apply eqn Ⓐ.

$2 C_1 C_3 + 0 = 100$

$2 C_1 C_3 = 100$

$\boxed{\therefore C_1 C_3 = 50}$

Substitituting these two condition in the above equation ~~u(x,t)~~ =

$u(x,t) = (C_1 x + C_2) C_3$ $\quad$ • $C_2 C_3 = 0$ • $C_1 C_3 = 50$

$u(x,t) = 50x$.

which shows that the temperature is independent of time. i.e, we take

$u_2(x) = 50x$ —— ③

Now, the temperature distribution in the rod is given by the solution the one dimensional heat transfer equation,

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t}$ and satisfying the initial condition 1, and boundary condition 2 unlike the general case (previous problem).

The boundary conditions are non-zero here.

So, the required solution is given by

$u(x,t) = u_1(x) + u_2(x,t)$. —— ④

(∵ temp is not zero) (∵ thus result)

where $u_1 x$ is the solution of the one dimensional heat equation $\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \times \frac{\partial u}{\partial t}$, involving 'x' only and satisfying the boundary condition 2, $u_2(x,t)$ is the transient nature of temperature as 'u' decreases, when 't' increases.

The eqn ④ is obtained by superimposing $u_1(x)$ & $u_2(x,t)$

$u_2(x,t) = u(x,t) - u_1(x)$ —— ⑤

Now we are putting ~~x=0~~

$x = 0$ in the above eqn ⑤

$u_2(0,t) = u(0,t) - u_1(0)$ (by condition-2) & by condition 3

$\Rightarrow u_2(0,t) = 0 - 0$

⇒ $u_2(0,t) = 0$ ——— ⑥

putting $x=2$, in eqn ⑥

$u_2(2,t) = u(2,t) - u_1(2)$.

⇒ $u_2(2,t) = 100 - 100$

⇒ $u_2(2,t) = 0$ ——— ⑦

Now we are putting $t=0$ in eqn ⑤

$u_2(x,0) = u(x,0) - u_1(x)$ [eqn 1, eqn 3]

⇒ ~~$u_2(x,0)$~~ = { ~~100x, 0<x<1~~ / ~~100, 1<~~

$$u_2(x,0) = \begin{cases} (100x - 50x), & 0<x<1 \\ (100 - 50x), & 1<x<2 \end{cases} \quad ——— ⑧$$

where equation ⑥ & ⑦ are our zero boundary conditions & eqn ⑧ is our initial temperature distribution.

This give, the initial temperature distribution $u_2(x,t)$. Now we can write $u_2(x,t) = \underbrace{(c_1 \cos px + c_2 \sin px)\, c_3\, e^{-hp^2 t}}$ ⑨

Now using the zero boundary conditions ie, eqn ⑥ & ⑦ in eqn ⑨, we get, the two condition

$$\boxed{c_1 = 0 \quad \& \quad p = \frac{n\pi}{L} \text{ where } (n = 1, 2, 3, \ldots) \qquad p = \frac{n\pi}{2}}$$

in the above eqn ⑨ we get, & using the principle of superposition we get

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \; e^{-h\cancel{p}\frac{n^2\pi^2}{4} t} \quad \text{———} ⑩$$

Now we are applying, the initial conditions in eqn ⑩

$$u_2(x,0) = \begin{cases} (100x - 50x), & 0 < x < 1 \\ (100 - 50x) & 1 < x < 2 \end{cases}$$

(t=0)

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} \times 1$$

$$= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{2} \quad \text{———} ⑪$$

Clearly this is a Fourier half range sine series

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

$$= \frac{2}{2}\int_0^2 u_2(x,0) \cdot \sin \frac{n\pi x}{2} \, dx.$$

$$= \left[\int_0^1 (100x - 50x) \sin \frac{n\pi x}{2} dx + \int_1^2 (100 - 50x) \sin \frac{n\pi x}{2} dx\right.$$

$$= 50\left[\int_0^1 x \sin \frac{n\pi x}{2} dx + \int_1^2 (2 - x) \sin \frac{n\pi x}{L} \, dx\right]$$

$$= 50\left[x\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - \left[(1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right]_0^1 + \right.$$

$$\left.\left[(2-x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (-1)\left[\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right]_1^2\right]$$

$$= 50\left[\frac{-2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right] +$$

$$\left[-\left(\frac{-2}{n\pi}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right]$$

$50 \times 2 \times \frac{4}{n^2\pi^2} \sin n\pi/2 = \frac{400}{n^2\pi^2} \sin n\pi/2$

$$= \frac{400}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad \text{—— (12)}$$

Here the transient nature of temperature is given by eqn ⑩ where Bn is given by (12)

Now, the temperature distribution function u(x, t) is given by the eqn ⑤. then i.e, by substituting the eqn ④ ~~ie by~~ & eqn ⑬ in the eqn. ⑤

$$u_2(x,t) = \sum_{n=1}^{\infty} \frac{400}{n^2\pi^2} \sin\frac{n\pi}{2} \sin\frac{n\pi x}{2} \cdot e^{-\frac{hn^2\pi^2 t}{4}} \qquad \text{—— (13)}$$

∴ The temperature distribution function

$$u(x, t) = u_1(x) + u_2(x, t)$$

23/2/16

② The end A and B of the rod of length 'L' are maintained at temperatures ~~100°c &~~ 0°c & 100°c respectively. Untill the steady state conditions ~~[illegible]~~ prevails. Suddenly ~~at~~ the temperature at the end A, increased to 20°c & temperature at the end 'B' is decreasing to 60°c. Find the temperature distribution in the at time 't'.

Ans: The temperature distribution $u(x,t)$ of the one dimensional heat equation is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h}\frac{\partial u}{\partial t} \quad \text{——} \; ①$$

In the steady state the heat flow is independent of time for which $\frac{\partial u}{\partial t} = 0$

Then the eqn ① reduces to ~~∂^2~~

$$\frac{\partial^2 u}{\partial x^2} = 0$$

SE $D^2(u) = 0$

AE $\lambda^2 = 0$

$\lambda = 0, 0$

$\Rightarrow u = A x + B$ —— ② where $C_1 = A$, $C_2 = B$ $(C_1 x + C_2) e^{0x}$

Now we are going to find the initial temperature $u(x, 0)$ by applying the first set boundary condition ie

$u = 0$, when $x = 0$ &

$u = 100$ when $x = L$ in eqn ②

we get,

$u = 0$, when $x = 0$

$0 = A \times 0 + B.$

$\Rightarrow \boxed{\therefore B = 0}$

$u = 100$, when $x = L$

$100 = A \times L + B.$

$\therefore B = 0.$

$A \times L + 0 = 100.$

$\boxed{\therefore A = \frac{100}{L}}$

Using these 2 conditions in the above eqn②

$u = \frac{100}{L} x + 0.$

$\Rightarrow \quad u = \frac{100}{L} x$

ie, $u(x) = \frac{100}{L} x.$ where $t = 0$

$u(x, 0) = \frac{100}{L} x$ ——— ③ it is the initial temperature distribution

suddenly the boundary condition are changed i.e

$u(0, t) = 20$, & $u(L, t) = 60$ ——— ④

Here the boundary conditions are non-zero. Hence (proceeding as in the previous case)

$u(x, t) = u_1(x) + u_2(x, t)$ ——— ⑤

where $u_1(x)$ is the steady state temperature distribution. ie, (independent of time) &

$u_2(x,t)$ is the transient nature of temperature where 'u' decreases with increase of time. i.e,

$u_2(x,t)$ is the solution of eqn ① & is given by ~~$u_2(x,t)=$~~

$$u_2(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-hp^2t} \quad ⑥$$

How to determine the $u_1(x)$ as follows.

In the steady state we have,

$u_1 = A_1 x + B_1$

i.e, $u_1(x) = A_1 x + B_1$

Now we are applying the 2nd boundary condition. i.e,

$u = 20$, when $x = 0$ & $u = 60$, when $x = L$

$u_1 = 20$, when $x = 0$

$20 = B_1$

$\boxed{\therefore B_1 = 20}$

$u_1 = 60$, when $x = L$

$60 = A_1 \times L + B_1$

$B_1 = 20$

$A_1 \times L + 20 = 60$

$\boxed{A_1 = \frac{40}{L}}$

Substituting the above two conditions

$u_1(x) = \frac{40}{L} x + 20$ ⑦

Now we are going to find $u_2(x)$.

From eqn ⑤ we have .

$u_2(x,t) = u(x,t) - u_1(x)$ ⑧

Now we are putting $x = 0$, in eqn ⑧

$u_2(0,t) = u(0,t) - u_1(0)$

$u_2(0,t) = 20 - 20$ (from eqn ④ & ⑦)

$\therefore u_2(0,t) = 0.$ ⑨

Next putting $x = L$ in the above eqn ⑧

$u_2(L,t) = u(L,t) - u_1(L)$

$u_2(L,t) = 60 + 60$ (fm eqn ④ & ⑦)

$u_2(L,t) = 0$ ⑩

Now we are resetting the initial condition by putting $t = 0$ in eqn ⑧

$u_2(x,0) = u(x,0) - u_1(x)$

$u_2(x,0) = \frac{100}{L} x - \left(\frac{40}{L} x + 20\right)$

$= \frac{100}{L} x - \frac{40}{L} x - 20$

$u_2(x,0) = \frac{60}{L} x - 20$ ⑪

Now we are applying the above two conditions in eqn ⑥ and also applying the principle of superposition we get

$$u_2(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\, e^{-\frac{hn^2\pi^2}{L^2}t} \quad \text{—⑫}$$

Now we are applying the initial condition no ⑪ in eqn ⑫

(t=0)

$$u_2(x,0) = \frac{60}{L}x - 20 = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} \times 1$$

$$= \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$$

which is clearly a Fourier sine series Where Bn is given by

$$B_n = \frac{2}{L}\int_0^L f(x) \cdot \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\int_0^L u_2(x,0)\, \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L}\int_0^L \left(\frac{60x}{L} - 20\right) \sin\frac{n\pi x}{L}\, dx$$

$$= \frac{2}{L}\left[\left(\frac{60}{L}x - 20\right)\left(\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right) - \left(\frac{60}{L}\right)\left(\frac{-L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right)\right]_0^L$$

$$= \frac{2}{L}\left[-40\,\frac{L}{n\pi}\cos n\pi - \left(\frac{L}{n\pi}\cdot 20\right)\right]$$

$$= \frac{-40}{n\pi}\left[2\cos n\pi + 1\right]$$

$$= \frac{-40}{n\pi}\left[1 + 2(-1)^n\right]$$

ie, then the eqn ⑫ become

$$u_1(x,t) = \sum_{n=1}^{\infty} -\frac{40}{n\pi}\left(1+2(-1)^n\right) \sin\frac{n\pi x}{L}\, e^{-\frac{hn^2\pi^2}{L}t}$$

$$= -\frac{40}{\pi}\sum_{n=1}^{\infty}\left(\frac{1+2(-1)^n}{n}\right) \sin\left(\frac{n\pi x}{L}\right) * e^{-\frac{hn^2\pi^2}{L^2}t}$$

∴ The temperature distribution $u(x,t)$ of the rod is given by eqn ⑤

ie, $u(x,t) = u_1(x) + u_2(x,t)$

ie $u(x,t) = \left(\frac{40}{L}x + 20\right) - \frac{40}{\pi}\sum_{n=1}^{\infty}\left(\frac{1+2(-1)^n}{n}\right) \sin \frac{n\pi x}{L}\Big) * e^{-\frac{hn^2\pi^2}{L^2}t}$

MPORTAN
QUS

? If the solution of one dimensional heat flow equation depends on the fourier cosine series, what would have been the nature of the end condition.

ns: The solution of one dimensional heat flow equation depends on the fourier cosine series if ~~the~~ ~~two~~ either the two end points are insulated or when the left end point is insulated and the right end ~~point~~ is kepted zero temperature. ~~and the right end is insulated when left is kepted zero temperature.~~

Q9 In the steady state condition derive the solution of one dimensional heat eqn?

Ans; The one dimensional heat eqn is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h} \frac{\partial u}{\partial t}$$

In the steady state the above eqn is independent of time. i'e,

$$\frac{\partial u}{\partial t} = 0$$

then the eqn ① become $\frac{\partial^2 u}{\partial x^2} = 0$

Since at this moment there is only one independent variable case. i'e the PDE changed to O.D.E. i.e

$$\frac{\partial^2 u}{\partial x^2} = 0$$

SF ⇒ $D^2 u = 0$

AE ⇒ $\lambda^2 = 0$

$\lambda = 0, 0$

.·.Y, $u = Ax + B$

⇒ $u(x) = Ax + B$

Insulated → means

There is not heat flow in that point

25/4/16
Monday

Insulated Qus(s)

temperature at one end of a bar of length 50cm is kepted one side at ~~zero~~ 0°C and other side at 100°C, with the sides are insulated. Until the steady state condition prevails. The two ends are ~~and~~ then suddenly insulated. Find the temperature distribution function?

ns:

The temperature distribution of a 1 dimensional heat transfer equation is

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h}\frac{\partial u}{\partial t} \quad \text{———} ①$$

[For solving a one dimensional heat transfer eqn two boundary condition & one initial condition is must but, in this problem, there is no initial condition is given but there is a hint is given i.e, Untill the steady state condition prevails]

Since, the rod is given a steady steate condition. i.e, the temperature distribution function is independent of time, i.e we get, $u = Ax + B$.

$$u(x) = Ax + B. \quad \text{———} ②$$

For finding the initial temperature condition now we are applying the initial boundary condition.

$u=0$, when $x=0$ $\quad$ $u=100$ when $x=50$

$0 = A\times 0 + B$ $\quad$ $100 = 50A + B.$

$\Rightarrow \boxed{B=0}$ $\quad$ $B=0 \Rightarrow 100 = 50A.$

$\therefore \boxed{A=2}$

Now applying the above obtained fun condition in eqn. ②

$$u(x) = 2x + 0.$$

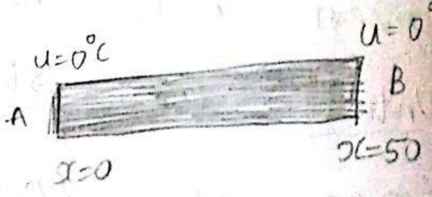
$$u(x) = 2x. \qquad (\text{where, } t=0)$$

i.e which is independent of time 't'

∴ The initial temperature function (condition),

$$u(x,0) = 2x. \text{ ——— } ③$$

Suddenly the both ends are insulated i.e, There is no heat flow at both ends i.e, The temperature distribution function $\left(\frac{\partial J(u)}{\partial x} = 0\right)$ $\boxed{\text{or}}$ the temperature gradient ⓞⓡ the heat flux is zero at both ends.



Then the boundary conditions are,

i.e,

$$\frac{\partial u(0,t)}{\partial x} = 0 \text{ ——— } ④$$

$$\frac{\partial}{\partial x} u(50,t) = 0 \text{ ——— } ⑤$$

i.e. the general solution of the eqn ① becomes

$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-hp^2t}$ ——— ⑥

Now we are going to apply the eqn ④&⑤ ie the boundary condition in ③. For that first we are differentiating the eqn ⑥ partially w.r. to x.

$$\frac{\partial}{\partial x} u(x,t) = p\, c_3\, e^{-hp^2t}\left[-c_1 \sin px + c_2 \cos px\right]$$ ——— ⑦

Now we are applying the boundary condition ④ in eqn ⑦.

$(x=0)$

$$0 = p c_3 e^{-hp^2t}\left[-c_1 \times 0 + c_2 \times 1\right]$$

$\therefore\ p\, c_2 c_3\, e^{-hp^2t} = 0$

$c_3 e^{-hp^2t} \neq 0$
p is eigen funct $\neq 0$
$\therefore c_2 = 0$

$\Rightarrow \boxed{c_2 = 0}$

Again we are applying the boundary condition ⑤ in eqn ⑦.

$c_2 = 0$

$(x=50)$

$$0 = p c_3 e^{-hp^2t}\left[-c_1 \sin 50p + \overset{0\times}{\cancel{c_2}} \cos 50p\right]$$

$\Rightarrow -p\, c_1 c_3 \sin 50p\, e^{-hp^2t} = 0.$

$p \neq 0$
$c_3 \neq 0$
$c_1 \neq 0$
$e^{-h} \neq 0$

$\Rightarrow \sin 50p = 0.$

ie, $\sin 50p = \sin n\pi$

$50p = n\pi \Rightarrow \boxed{p = \frac{n\pi}{50}}, \ n = 0, 1, 2 \dots$

[because $n = 0, 1, 2$ both sides insulated]

Now we are substituting the above obtained two condition.

ie $C_2 = 0$ & $p = \frac{n\pi}{50}$, $n = 1, 2 \ldots$

in equation ⑥, we get.

$$u(x,t) = \left[C_1 \cos \frac{n\pi}{50} x + 0 \times \sin \frac{n\pi}{50} x\right] C_3 e^{-h\frac{n^2\pi^2}{2500}t}$$

$$\Rightarrow u(x,t) = C_1 C_3 \cos \frac{n\pi}{50} x \times e^{-h\frac{n^2\pi^2}{2500}t} \quad (n = 0, 1, 2, 3$$

and also, applying principle of super position we get;

if C_1 is here = An
if not C_1 = Bn

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi}{50} x \times e^{-h\frac{n^2\pi^2}{2500}t} \quad (n = 0, 1, 2, \ldots) \text{ ——— ⑧}$$

where $\underline{A_n = C_1 C_3}$

Now we are applying the initial condition ③ ie,

ie $u(x, 0) = 2x$. in the eqn ⑧
ie, $(t=0)$

$n = 0, (A_0 \times 1) + \sum_{n}$

$$u(x,0) = 2x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi}{50} x \times 1$$

$$2x = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{50}$$

so, The above eqn. can be rearranged as

$$2x = \underline{a_0} + \sum^{\infty} a_n \cos n\pi x$$

which is clearly a half range cosine series & $A_0 = \frac{a_0}{2}$, $A_n = a_n$

Then, ~~$A_0 =$~~

$$\frac{a_0}{2} = A_0 = \frac{1}{L}\int_0^L f(x)\,dx.$$

$$= \frac{1}{50}\int_0^{50} 2x\ dx.$$

$$= \frac{1}{50}\left(2\frac{x^2}{2}\right)_0^{50}$$

$$= \frac{1}{50}\left(x^2\right)_0^{50}$$

$$= \frac{1}{50}(2500-0) = \underline{\underline{50}}.$$

$$a_n = A_n = \frac{2}{L}\int_0^L f(x)\ \cos\frac{n\pi x}{L}\,dx.$$

$$= \frac{2}{50}\int_0^{50} 2x.\ \cos\frac{n\pi x}{50}\,dx$$

$$= \frac{1\times 2}{25}\left(x\times\frac{\sin\frac{n\pi x}{50}}{n\pi/50}\right)_0^{50} - \left(1\times\frac{-\cos\frac{n\pi x}{50}}{(n\pi/50)^2}\right)_0^{50}$$

$$= \frac{2}{25}\left(\frac{50x}{n\pi}\sin\frac{n\pi x}{50}\right)_0^{50} + \left(\frac{50^2}{n^2\pi^2}\cos\frac{n\pi x}{50}\right)_0^{50}$$

$$= \frac{2}{25}\left(0 + \left(\frac{50^2}{n^2\pi^2}\cos\frac{n\pi\times 50}{50} - \frac{50^2}{n^2\pi^2}\right)\right)$$

$$= \frac{2}{25}\left[\frac{50^2}{n^2\pi^2}(-1)^n - \frac{50^2}{n^2\pi^2}\right]$$

$$= \frac{2 \times 50^2}{25 \times n^2 \pi^2} \left[(-1)^n - 1\right]$$

25

5000

$\frac{5000}{25} = 200$

$$= \frac{200}{n^2\pi^2} \left[(-1)^n - 1\right]$$

$$An = \begin{cases} \frac{-400}{n^2\pi^2}, & n \text{ is odd}, \\ 0, & n \text{ is even}, \end{cases}$$

$$u(x,t) = 50 - \frac{400}{\pi^2} + \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi}{50}$$

$$\times e^{-h\left\{\frac{(2n-1)^2 \pi^2}{2500}\right\}t}$$

(2) Find the temperature distribution in a bar of length π whose surface is thermally insulated. with end points maintain at 0°c. The initial temperature distribution in the rod is given by

$$u(x,0) = \begin{cases} x, & 0 \le x \le \pi/2 \\ \pi - x, & \pi/2 \le x \le \pi \end{cases}$$

The temperat distribution fauh u

$u = 0$

subject to the boundary condition

$u(0,t)=0$ ——— ②

$u(\pi,t)=0$ ——— ③

the given initial temperature condition, ④

$$u(x,0)=\begin{cases} x, & 0\le x\le \pi/2 \\ \pi-x, & \pi/2\le x\le \pi. \end{cases}$$

By using the method of separation of variable the solution of eqn ① is,

The general solution

$$u(x,t)=(c_1\cos px+c_2\sin px)\,c_3e^{-hp^2t}. \quad ⑤$$

Now we are applying the given boundary condition ② & 3 in the general solution ⑤

we get. $\boxed{c_1=0,}$ $\boxed{p=\frac{n\pi}{L}=\frac{n\pi}{\pi}=\underline{n}}$ $[n=0,1,2.\ldots]$

Now substituting the above 2 conditions in eqn ⑤

$$u(x,t)=[0\times\cos nx+c_2\sin nx]\,c_3e^{-hn^2t}.$$

$$u(x,t)=c_2c_3\sin nx\,e^{-hn^2t} \quad (n=0,1,2\ldots)$$

& also applying the principle of superposition we get.

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin nx\,e^{-hn^2t} \quad ⑥$$

$n=0,1,2\ldots$

where $\underline{c_2c_3=B_n}$

Now we are applying the initial condition 4 in eqn ⑥

$(t=0)$

$$u(x,0)=\begin{cases} x, & 0\le x\le \pi/2 \\ \pi-x, & \pi/2\le x\le\pi \end{cases}=\sum_{n=1}^{\infty}B_n\sin nx\times 1$$

clearly this is fourier half range sine series where $B_n = \frac{2}{L}\int_0^L f(x) \cdot \sin\frac{n\pi x}{L}\, dx.$

$$B_n = \frac{2}{\pi}\int_0^{\pi} u(x,0)\,\sin\frac{n\pi x}{\pi}\, dx.$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \,\cancel{dx}\, \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\,\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left(x \times -\frac{\cos nx}{n}\right)_0^{\pi/2} - \left(1 \times -\frac{\sin nx}{n^2}\right)_0^{\pi/2}\right]_0^{\pi/2} +$$

$$\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\times\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{\pi}\Big[\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\Big) + \Big\{-\Big(\frac{-\pi}{2n}\times\cos\frac{n\pi}{2} - \frac{1}{n^2}\sin\frac{n\pi}{2}\Big]$$

$$= \frac{4}{\pi n^2}\sin\left(\frac{n\pi}{2}\right)$$

Hence from the eqn⑥ the required solution is

$$u(x,t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right)\sin(nx)\,e^{-hn^2 t}$$

(3) Find the temperature distribution function of a rod of length π which is totally insulated including the n' and the initial temperature distribution is 100 cos x.

[illegible] insulated (totally)

Ans: The temperature distribution function u(x,t) of wave transfer eqn is given by

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h}\frac{\partial u}{\partial x} \quad \text{——} ①$$

since, The both ends of the rod are thermally insulated.

∴ the boundary conditions are

$$\frac{\partial}{\partial x}\{u(0,t)\} = 0 \quad \text{——} ②$$

$$\frac{\partial}{\partial x} u(\pi,t) = 0 \quad \text{——} ③$$

and the initial temperature condition is

$$u(x,0) = 100 \cos x \quad \text{——} ④$$

Then by the method of separation of variable the physically acceptable solution is

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-hp^2 t} \quad \text{——} ⑤$$

Now we are going to apply the boundary condition ② & ③ in eqn ⑤. For that first we are differentiating the eqn ⑤ partiall with respect to x.

ie

$$u(x,t) = p c_3 e^{-hp^2 t}\left[-c_1 \sin px + c_2 \cos px\right] \text{——} ⑥$$

Now we are applying the boundary condition 2 in eqn ⑥

$x=0$

⇒ $0 = p c_3 e^{-hp^2 t} [-c_1 \times 0 + c_2 \times 1]$

⇒ $p c_2 c_3 e^{-hp^2 t} = 0$

⇒ $\boxed{c_2 = 0}$

Applying the boundary condition ③ in eqn ⑥.

$c_2 = 0]$

$x = \pi$

$0 = p c_3 e^{-hp^2 t}[- c_1 \sin p\pi + \overset{0}{\cancel{c_2}} \cos p\pi]$

$c_2 = 0$

$-p c_1 c_3 e^{-hp^2 t} \sin p\pi = 0$

⇒ $\sin p\pi = 0$

⇒ $\sin p\pi = \sin n\pi$

⇒ $\boxed{p = \frac{n\pi}{\pi} = n.}$ $n = 0, 1, 2 \ldots$

So, substituting these values in eqn ⑤ we get,

$u(x,t) = (c_1 \cos nx + \overset{0}{\cancel{c_2}} \sin nx)\, c_3 e^{-hn^2 t}$

$u(x,t) = c_1 c_3 \cos nx\, e^{-hn^2 t}$ $(n = 0, 1, 2 \ldots)$

$(c_1 c_3 = A_n)$

Then by applying the principle of superposition we get,

$u(x,t) = \sum_{n=0}^{\infty} A_n \cos nx\, e^{-hn^2 t}$ ——— ⑦

Also $t=0$ applying the initial condition in eqn ⑦

$$u(x,0) = 100\cos x = \sum_{n=0}^{\infty} A_n \cos nx \, e^{-hn^2t} \times 1$$

$n=1$

$$\Rightarrow 100 \cos x = \sum_{n=0}^{\infty} A_n \cos nx.$$

this gives $A_1 = 100$ when $n=1$ & all other $n=0$. Hence from Eqn ⑦ the required solution,

$$u(x,t) = 100 \cos x \, e^{-ht}$$

(4) Q. The temperature at one end of a bar of length 'l' cm. with insulated sides is kepted $0°c$. & other is kepted $100°c$. Untill the steady state condition prevails. The two ends are then suddenly insulated. Find the temperature distribution in the bar.

Ans: The temperature distribution $u(x,t)$ of a one dimensional heat transfer eqn is given by $\frac{\partial^2 u}{\partial x^2} = \frac{1}{h}\frac{\partial u}{\partial t}$ ——①

Since, the rod is in the steady state. we have $u = Ax + B$

$$u(x) = Ax + B \text{ ——— ②}$$

Now here we are applying the initial

boundary condition

u = 0
when $x = 0$

$0 = A \times 0 + B$

$\Rightarrow \boxed{B = 0}$

u = 100
when $x = L$

$100 = A \times L + B$

$B = 0$

$100 = A \times L$

$\therefore \boxed{A = \frac{100}{L}}$

Then we are applying the above obtained two condition.

i.e, $B = 0$ & $A = \frac{100}{L}$ in eqn ②
the initial condition as

$$u(x,t) = \frac{100}{L} x + 0$$

$$u(x,0) = \frac{100}{L} x,$$

This eqn is independent of t.
i.e the initial temperature distribution

$$u(x,0) = \frac{100}{L} x. \quad \text{———} ③$$

suddenly the two ends are insulated.
i.e, the boundary conditions becomes.

$$\frac{\partial}{\partial x} u(0,t) = 0 \quad \text{———} ④$$

$$\frac{\partial}{\partial x} u(L,t) = 0 \quad \text{———} ⑤$$

Now by the method of separation of variable the general solution of a heat transfer eqn. ie here eqn ① becomes

$$u(x,t) = (c_1 \cos px + c_2 \sin px)\, c_3 e^{-hp^2t} \quad ⑥$$

Now we are going to apply the zero boundary condition eqn ④ & ⑤ in eqn ⑥. For that first we are differentiating the eqn. ⑥ partially w. r. to x.

ie,

$$\frac{\partial}{\partial x} u(x,t) = pc_3 e^{-hp^2t}(-c_1 \sin px + c_2 \cos px) \quad ⑦$$

Now we are applying the ~~first~~ ~~zero~~ ~~bou~~ ② zero boundary condition eqn ④ & ⑤ in eqn ⑦.

we get, $\boxed{c_2 = 0}$ & $\boxed{P = \frac{n\pi}{L},\ \cancel{n} = 0, 1, 2 \ldots}$

substituting these two values in eqn ⑥ and also including the principle of superposition we get

$(A_n = c_1 c_3)$

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{L}\, e^{-h\frac{n^2\pi^2}{L^2}t} \quad ⑧$$

ie, $(n=0)$

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L}\, e^{-h\frac{n^2\pi^2}{L^2}t} \quad \cancel{⑧}\ ⑨$$

Finally we are applying our initial condition in eqn ⑨

$$u(x,0) = \frac{100}{L}x \quad \text{— ⑬.}$$

$$\frac{100}{L}x = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} \quad \times 1$$

$\frac{a_0}{2} A_0$

cosine

$A_0 = \frac{2}{L}\int$ ×

$A_0 = \frac{1}{L}\int$

$\left[\text{This is of the form, } \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \right]$

Clearly this a fourier cosine series

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx.$$

$$= \frac{1}{L}\int_0^L \frac{100}{L}x\,dx.$$

$$= \frac{100}{L\times L}\left(\frac{x^2}{2}\right)_0^L$$

$$= \frac{100}{L^2}\,\frac{L^2}{2} = \underline{\underline{50}}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\,\cos\frac{n\pi x}{L}\,dx.$$

$$= \frac{2}{L}\int_0^L \frac{100}{L}x.\ \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\times\frac{100}{L}\left[x.\ \sin\frac{n\pi x}{L}\right]_0^L - \left(1 \times -\frac{\cos n\pi x}{L}\right)_0^L$$

$$= \frac{2}{L}\times\frac{100}{L}\left[\left(L\times\frac{\sin n\pi}{n\pi/L}\right) + \frac{\cos(n\pi)}{(n\pi/L)^2}\right]^L - \frac{1}{\left(\frac{n^2\pi^2}{L^2}\right)}$$

$\cancel{\frac{200}{L^2} \times \frac{(-1)^n}{n^2\pi^2/L^2}} \quad = \frac{-200}{L2} \quad \cos$

$\cancel{\frac{200}{n^2\pi^2}} \times (-1)^n.$

$$= \frac{200}{L2}\left(\frac{(-1)^n}{n^2\pi^2/L^2} - \frac{1}{n^2\pi/L^2}\right)$$

$$= \frac{200}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$= \begin{cases} -\frac{400}{n^2\pi^2}, & n \text{ is odd} \\ 0, & n \text{ is eve.} \end{cases}$$

Hence ~~From eqn ⑨~~ the G.S of the rod is given by eqn ⑨

$$u(x,t) = 50 + \frac{200}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n2} \times \cos\frac{n\pi x}{L} \times e^{-\frac{hn^2\pi^2}{L^2}t}$$